em-que-02057

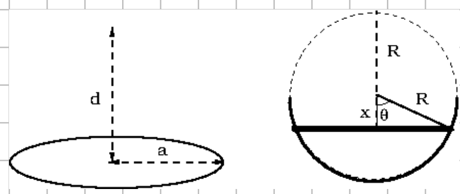
Question: Find the field due a hemisphere at the north pole
Assume uniform charge density ρ.

The electric field due to a circular disk of radius a at a point above the center at distance d is

$$E = \frac{Q}{2\pi\epsilon_0 a^2}\left(1 - \frac{d}{\sqrt{a^2+d^2}}\right)$$

where Q is the total charge on the disk.

Divide the hemisphere into disks by parallel planes as shown in the figure. Consider one such disk at a distance x from the center.



$$x = R\cos\theta \qquad dx = -R\sin\theta\, d\theta$$

Radius of the disk $= R\sin\theta \equiv a$

Volume of the disk $= (\pi a^2)\, dx$

$$= \pi R^2 \sin^2\theta \, R\sin\theta\, d\theta$$

Total charge on the disk

$$= \rho(\pi R^3 \sin^3\theta\, d\theta) \equiv dQ$$

Distance of the field point from the center of the disk $= (R + R\cos\theta) \equiv z$

$\therefore$ Field due to the disk between x and $x+dx$

$$dE = \frac{dQ}{2\pi\epsilon_0 a^2}\left(1 - \frac{d}{\sqrt{a^2+d^2}}\right) \text{ along z axis}$$

$$\cdot\, a^2 + d^2 = (R+R\cos\theta)^2 + R^2\sin^2\theta = R^2 + R^2\cos^2\theta + 2R\cos\theta + R^2\sin^2\theta$$

$$= 2R^2 + 2R\cos\theta$$

$$= 2R^2(1+\cos\theta)$$

Total charge $Q = \frac{2}{3}\pi R^3 \rho$

$$dE = \frac{\rho(\pi R^3 \sin^3\theta\, d\theta)}{2\pi\epsilon_0 R^2\sin^2\theta}\left(1 - \frac{(1+R\cos\theta)}{\sqrt{2R^2(1+\cos\theta)}}\right) d\theta$$

$$= \frac{\rho R}{2\epsilon_0}\sin\theta\left(1 - \sqrt{(1+\cos\theta)/2}\right)$$

Therefore total field on the north pole

$$E = \left(\frac{\rho R}{2\epsilon_0}\right)\left[\int_0^{\pi/2}\sin\theta - \int_0^{\pi/2}\sin\theta\sqrt{(1+\cos\theta)/2}\right]$$

change variable $t = 1+\cos\theta$

$$= \left(\frac{\rho R}{2\epsilon_0}\right)\left[-\cos\theta\Big|_0^{\pi/2} - \int_1^2 dt\,\sqrt{t/2}\right]$$

$$= \left(\frac{\rho R}{2\epsilon_0}\right)\left[1 - \frac{2}{3}\frac{t^{3/2}}{\sqrt{2}}\Big|_0^1\right] = \left(\frac{\rho R}{2\epsilon_0}\right)\left[1 - \frac{\sqrt{2}}{3}(t^{3/2})\Big|_1^2\right]$$

$$= \left(\frac{\rho R}{2\epsilon_0}\right)\left[1 - \frac{4}{3} + \frac{\sqrt{2}}{3}\right] = \left(\frac{\rho R}{6\epsilon_0}\right)(\sqrt{2} - 1)$$