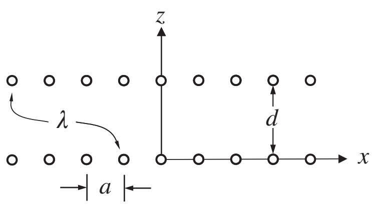

Figure 7.7: Edge view of a "parallel-plate" Faraday cage composed of two parallel arrays of infinitely long charged lines aligned parallel to the y-axis. Each line has a uniform charge per unit length λ. The diagram repeats periodically out to $x = \pm\infty$. The two walls of the cage are separated by a distance d. Within each wall, the charged lines are separated by a distance a.

Faraday's *diary*, January 15 1836

Have been for some days engaged in building up a cube of 12 feet in the side. It consists of a slight wooden frame held steady by diagonal ties of cord; the whole being mounted on four glass feet. The sides, top and bottom are covered in paper. The top and bottom have each a cross framing of copper wire connected by copper wires passing down the four corner uprights. The sheets of paper which constitute the four sides have each two slips of tin foil pasted on their inner surface. These are connected below with copper wire so that all the metallic parts are in communication. Access to the inside was made by cutting a flap in the paper. Connecting this cube by a wire with the Electrical Machine, I can quickly and well electrify the whole. I now went in the cube standing on a stool and [my assistant] Anderson worked the machine until the cube was fully charged. I could by no appearance find any traces of electricity in myself or the surrounding objects. In fact, the electrification without produced no consequent effects within.

7.5.1 Faraday's Cage

The diary entry just above describes a wire mesh "cage" which effectively shields its interior from an electrostatic potential applied to its surface. The physics of the shielding can be understood using the "parallel-plate" cage shown in Figure 7.7, where an infinite number of infinitely long and uniformly charged lines lie in two parallel planes. This is a good model for a grid of conducting wires because, if every line carries a charge per unit length λ, the potential at a distance $s \ll a$ from any particular line (regarded as the origin) is

$$\varphi(s) = -\frac{\lambda}{2\pi\epsilon_0}\ln|s| + const. \tag{7.44}$$

The equipotentials of (7.44) are cylinders so the charged lines are electrostatically equivalent to an array of conducting wires held at a common potential.

To find the potential between the "plates" of the cage, we begin with the potential produced by the lower grid of wires alone. The reflection symmetry of the grid implies that this two-dimensional potential must obey $\varphi(-x, -z) = \varphi(x, z)$. Moreover, $\varphi(x, z) \to 0$ as $|z| \to \infty$ because the source charge is localized in the z-direction. This tells us to choose the sign of the separation constant so $Z(z)$ in (7.27) decays exponentially away from the grid in both directions. Putting all this information together yields a separated-variable solution for the lower grid of wires with the form

$$\varphi(x, z) = A + B|z| + \sum_\gamma C_\gamma \cos(\gamma x)\exp(-\gamma|z|). \tag{7.45}$$

It is essential that we retain the term proportional to $|z|$ [the $\gamma = 0$ term in (7.27)] because, when $|z| \gg a$, the grid is indistinguishable from an infinite charged sheet. Indeed, precisely this fact tells us that $A = 0$ and

$$B = -\lambda/2a\epsilon_0. \tag{7.46}$$

The geometry of the wires imposes the *periodicity* condition $\varphi(x + a, z) = \varphi(x, z)$. This means that the separation constant γ takes the discrete values $\gamma = 2\pi m/a$ with $m = 1, 2, \ldots$ Accordingly,

$$\varphi(x, z) = -\frac{\lambda}{2a\epsilon_0}|z| + \sum_{m=1}^{\infty} C_m \cos(2\pi mx/a) \exp(-2\pi m|z|/a). \tag{7.47}$$

We fix the expansion coefficients C_m by imposing the matching condition (7.4) in the form

$$\left.\frac{\partial\varphi}{\partial z}\right|_{z=0^-} - \left.\frac{\partial\varphi}{\partial z}\right|_{z=0^+} = \frac{\sigma(x)}{\epsilon_0}. \tag{7.48}$$

The relevant $\sigma(x)$ is the surface charge density of the array of charged lines,

$$\sigma(x) = \lambda \sum_{p=-\infty}^{\infty} \delta(x - pa). \tag{7.49}$$

A brief calculation using the Fourier expansion[7]

$$\sum_{p=-\infty}^{\infty} \delta(x - 2\pi p) = \frac{1}{2\pi} + \frac{1}{\pi} \sum_{m=1}^{\infty} \cos mx \tag{7.50}$$

yields the potential created by the $z = 0$ wire mesh:

$$\varphi(x, z) = -\frac{\lambda}{2a\epsilon_0}|z| + \frac{\lambda}{2\pi\epsilon_0} \sum_{m=1}^{\infty} \frac{1}{m} \cos(2\pi mx/a) \exp(-2\pi m|z|/a). \tag{7.51}$$

The reader can confirm that (7.51) reproduces (7.44) in the plane of the grid.

The upper grid of wires contributes a potential identical to (7.51) except that $z \to z - d$ where d is the separation between the two grids. When we add the two together, the first term in (7.51) *cancels* the corresponding term in the upper grid potential when $0 < z < d$. The remaining, mesh-induced, contributions are *exponentially small* if $z \gg a$ and $d - z \gg a$. Therefore, the electric field inside the cage is essentially *zero* at all observation points that lie farther away from the cage walls than the spacing between the wires of the cage. The condition $d \gg a$ is a design prerequisite for all practical Faraday cages.

7.5.2 The Electrostatic Potential Has Zero Curvature

The duct potential (7.38) and the cage potential (7.51) are both composed of products of sinusoidal and exponential functions in orthogonal directions. This is a consequence of the constraint (7.24) imposed on the separation constants. Since Laplace's equation (7.5) is the origin of the constraint, we can use geometrical language and ascribe this behavior to the requirement that the total *curvature* of $\varphi(\mathbf{r})$ vanish at every point where $\nabla^2\varphi = 0$. This insight provides a qualitative understanding of Earnshaw's theorem[8] because it says that the one-dimensional curvatures of the potential along each of the three Cartesian directions cannot all have the same algebraic sign.

[7] See Example 1.6.

[8] Earnshaw's theorem states that $\varphi(\mathbf{r})$ cannot have a local maximum or a local minimum in a charge-free volume of space. See Section 3.3.3.