Poynting Theorem — Example of Flow of Energy

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Summary

In this section we prove the Poynting theorem about conservation of energy for system of charges and electromagnetic fields. Two important points emerge.

1. The fields have energy with energy density given by

$$U_{em} = \frac{\epsilon_0}{2}|\vec{E}|^2 + \frac{1}{2\mu_0}|\vec{B}|^2 \tag{1}$$

2. The energy lost or gained by an physical system, volume V, is to be viewed as flowing from the boundary of the system.

3. The rate of flow of energy through the boundary is given by the Poynting vector

$$S = \frac{1}{\mu_0}(\vec{E} \times \vec{B}), \tag{2}$$

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1 Energy momentum conservation — Poynting theorem

In section we will discuss conservation of energy for charges in interaction with electromagnetic fields. An expression for rate of doing work by electromagnetic forces will be derived. This will then lead us to the rate of change of mechanical energy. The resulting conservation law is a local conservation law. For more details see end of this section.

$$U_{em} = \frac{\epsilon_0}{2}|\vec{E}|^2 + \frac{1}{2\mu_0}|\vec{B}|^2 \tag{3}$$

It may be noted that the above form of energy per unit volume coincides with with sum of expression obtained for charge and current distributions for stationary charge and current distributions.

The new insight that we gain here is the possibility of exchange of energy between charges, currents and electromagnetic fields.

The resulting equation will be of the form of equation of continuity and is a local conservation law for energy for charges and electromagnetic fields. For mechanical systems, the

When a point charged particle moves in electromagnetic field, it experiences a force

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}). \tag{4}$$

For a continuous charge distribution, the force on a small volume elementdV is given by

$$\Delta\vec{F} = dq(\vec{E} + \vec{v} \times \vec{B}) = (\rho dV)(\vec{E} + \vec{v} \times \vec{B}). \tag{5}$$

We now wish to compute work done on charges present in a small volume ΔV. The displacement of the charge element in time Δt is $\vec{v}\Delta t$ and hence the work done by the electromagnetic forces in time Δt is $\Delta\vec{F} \cdot \vec{v}\Delta t$. Therefore, work done per sec on all charges is given by

$$\frac{dW}{dt} = \iiint_V \rho(\vec{E} + \vec{v} \times \vec{B}).\vec{v}\, dV. \tag{6}$$

By work energy theorem, this work done will be equal to the rate of change of mechanical energy (kinetic energy etc.) .

$$\begin{aligned} \frac{dW}{dt} &= \iiint_V \rho(\vec{E} + \vec{v} \times \vec{B}).\vec{v} dV. \\ &= \iiint_V \rho\vec{v} \cdot \vec{E} dV = \iiint_V \vec{j} \cdot \vec{E} dV \\ &= \iiint_V \frac{1}{\mu_0}\vec{E} \cdot (\nabla \times \vec{B}) dV - \epsilon_0 \iiint \vec{E} \cdot \frac{\partial \vec{E}}{\partial t}\, dV. \end{aligned} \tag{7}$$

where the Maxwell's fourth equation

$$\nabla \times \vec{B} = \mu_0\vec{j} + \epsilon_0\mu_0\frac{\partial \vec{E}}{\partial t} \tag{8}$$

has been used to write the current density, $\vec(j)$, in terms of $\vec{E}, \vec{B}$. Next we use the vector calculus identity

$$\nabla \cdot (\vec{E} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{E}) - \vec{E} \cdot (\nabla \times \vec{B}). \tag{9}$$

On use of the Maxwell's equation $\nabla \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$, Eq.(8) becomes

$$\begin{aligned} \vec{E} \cdot (\nabla \times \vec{B}) &= -\nabla \cdot (\vec{E} \times \vec{B}) - \vec{B} \cdot (\nabla \times \vec{E}) \\ &= -\nabla \cdot (\vec{E} \times \vec{B}) - \vec{B} \cdot \Big(\frac{\partial \vec{B}}{\partial t}\Big). \end{aligned} \tag{10}$$

Eq.(7) and (10)lead to the following equation

$$\begin{aligned} \frac{dW}{dt} &= \epsilon_0 \iiint\limits_V \vec{E} \cdot \frac{\partial \vec{E}}{\partial t}\, dV - \frac{1}{\mu_0} \iiint\limits_V \vec{B} \cdot \Big(\frac{\partial \vec{B}}{\partial t}\Big) - \frac{1}{\mu_0} \iiint\limits_V \nabla \cdot (\vec{E} \times \vec{B}) \\ &= \epsilon_0 \iiint\limits_V \vec{E} \cdot \frac{\partial \vec{E}}{\partial t}\, dV - \frac{1}{\mu_0} \iiint\limits_V \vec{B} \cdot \Big(\frac{\partial \vec{B}}{\partial t}\Big) - \frac{1}{\mu_0} \iiint\limits_S (\vec{E} \times \vec{B}) \end{aligned} \tag{11}$$

where S is the surface enclosing the volume V. Introducing the Poynting vector

$$S = \frac{1}{\mu_0} (\vec{E} \times \vec{B}), \tag{12}$$

and using Gauss divergence theorem, we rewrite Eq.(10) in the form

$$\frac{dW}{dt} = -\iiint\limits_V \frac{\partial}{\partial t} \left(\frac{\epsilon_0}{2} |\vec{E}|^2 + \frac{1}{2\mu_0} |\vec{B}|^2 \right) - \iint\limits_S \hat{n} \cdot \vec{S} dS. \tag{13}$$

or

$$\frac{d}{dt} \left[W + \iiint\limits_V \left(\frac{\epsilon_0}{2} |\vec{E}|^2 + \frac{1}{2\mu_0} |\vec{B}|^2 \right) \right] = -\iint\limits_S \hat{n} \cdot \vec{S} dS. \tag{14}$$

The expression

$$U_{em} = \frac{\epsilon_0}{2} |\vec{E}|^2 + \frac{1}{2\mu_0} |\vec{B}|^2 \tag{15}$$

is the energy density associated with the electromagnetic field. This expression already appears for energy density of static charge and current distributions.

✍ Eq.(15) has the interpretation that
change in energy of charges per sec + change in energy of e.m. field in volume V = flow of energy through the surface Sper sec, and the flow of energy per unit area per sec is given by the Poynting vector $\vec{S}$

Here the following points should not be missed.

- that the electromagnetic fields carry energy density
- electromagnetic fields can exchange energy energy with a mechanical systems
- energy conservation, just like charge conservation is a local conservation law

✍ You should think about it carefully, and ask yourself if these are new results not present in case of static fields.

References

[1] Sec 27-1 **Energy conservation and electromagentism** R. P. Feynman, Robert B. Leighton and Mathew Sands*Lectures on Physics*, vol-II, B.I. Publications (1964)

[2] Sec 8.1.2 **Poynting's Theorem** David Griffiths, *Introduction to Electrodynamics*, 3rd EEE edn, Prentice Hall of India Pvt Ltd New Delhi, (2002).

You May Also Be Interested In

A conservation law in Newtonian mechanics means that a physical quantity $G(\mathbf{q}((\mathbf{t}),\dot{\mathbf{q}})$ is constant along the physical trajectory. Thus when $\mathbf{q}(\mathbf{t})$ is a solution of equations of motion,G is independent of time.

$$\frac{dG}{dt} = 0$$

In special relativity the conservation laws must have a form treating space and time on equal footing. Thus they are always expressed as an equation of continuity

$$\frac{dG_0}{dt} - \nabla \cdot \vec{G} = 0. \tag{16}$$

where the four component object transforms $G = (G_0, \vec{G})$ like a four vector. All conservation laws must have this from.

Examples are charge conservation, and energy and momentum conservation etc.