# Energy Levels of Harmonic Oscillator Using Algebraic Methods
# Lectures given at University of Hyderabad

A. K. Kapoor
akkhcu@gmail.com, `http://webphys.uohyd.ernet.in`

August 22, 2013 Ver 1.0

**Abstract**

The energy levels of harmonic oscillator are derived using canonical commutation relations.

## Operator algebra for harmonic oscillator

The classical Hamiltonian for harmonic oscillator in one dimension is

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 \tag{1}$$

The corresponding operator $\hat{H}$ is obtained by replacing the position and momentum $x, p$ by the operators $\hat{x}, \hat{p}$ satisfying canonical commutation relation $[\hat{x}, \hat{p}] = i\hbar$. Note that we will not use any explicit representation of the operators. The commutation relation is sufficient to obtain all the answers.

We introduce operators $N, a, a^\dagger$ by

$$a = \frac{1}{\sqrt{2m\omega\hbar}}(\hat{p} - im\omega\hat{x}), \tag{2}$$

$$a^\dagger = \frac{1}{\sqrt{2m\omega\hbar}}(\hat{p} + im\omega\hat{x}), \tag{3}$$

$$N = a^\dagger a. \tag{4}$$

It is easy to see that these operators satisfy the following identities:

$$[a, a^\dagger] = 1, \qquad [N, a] = -a, \qquad [N, a^\dagger] = a^\dagger, \tag{5}$$

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 = (N + 1/2)\hbar\omega. \tag{6}$$

The last expression in (Eq.(6)) is obtained by expressing the operators $\hat{x}, \hat{p}$ appearing in $\hat{x}^2, \hat{p}^2$ in terms of $a, a^\dagger$, expanding the squares, *maintaining the order of operators carefully* and using the relations (Eq.(5)).

## Eigenvalues of $N$ are non negative

Let $\nu$ be an eigenvalue and $|\psi\rangle$ be the corresponding normalized eigenvector:

$$N|\psi\rangle = \nu|\psi\rangle. \tag{7}$$

Taking scalar product[1] with $|\psi\rangle$, we see that $\langle\psi|N|\psi\rangle = \nu$ is positive because

$$\langle\psi|N|\psi\rangle = \langle\psi|a^\dagger a|\psi\rangle = (a\psi, a\psi) = \|a\psi\|^2 \geq 0. \tag{8}$$

## The operator $a$ lowers the eigenvalues of $N$

Let $\nu$ be an eigenvalue of $N$ and $|\psi\rangle$ be the corresponding eigenvector as in (Eq.(7)). The operator $a$ act like lowering operator for the eigenvalues of $N$. The vector $|\phi_1\rangle$, obtained by applying $a$ on $|\psi\rangle$, is an eigenvector of $N$ with eigenvalue $\nu - 1$. To see this consider $N|\phi_1\rangle$

$$\begin{aligned} N|\phi_1\rangle &= Na|\psi\rangle = (aN - a)|\psi\rangle & (9)\\ &= (a\nu - a)|\psi\rangle = (\nu - 1)a|\psi\rangle & (10)\\ \therefore N|\phi_1\rangle &= (\nu - 1)|\phi_1\rangle. & (11) \end{aligned}$$

Hence $|\phi_1\rangle$, if non zero, is an eigenvector of $N$ with eigenvalue $\nu - 1$. Using this successively, we see that the result $a^n|\psi\rangle$, of applying $r$ powers of $a$ on $|\psi\rangle$ would give an eigenvector of $N$ with eigenvalue $\nu - r$:

$$N\,(\,a^r|\psi\rangle\,) = (\nu - r)\,(\,a^r|\psi\rangle\,). \tag{12}$$

Now there are two possibilities: $a^r|\psi\rangle = 0$ or else if the vector $a^r|\psi\rangle$ is non zero, it is an eigenvector of $N$ with eigenvalue $\nu - r$. since the eigenvalues of $N$ have to be non negative, we must have

$$a^r|\psi\rangle = 0 \text{ for all } r > \nu \tag{13}$$

Taking $m$ to be the maximum integer such that $m < \nu$ we see that

$$a^m|\psi\rangle \neq 0 \quad \text{and} \quad a^{m+1}|\psi\rangle = 0 \tag{14}$$

Using $|0\rangle$ to denote vector obtained from $(a^m|\psi\rangle)$ after normalization, we see that $|0\rangle$ is an eigenvector of $N$ with eigenvalue 0. To see, this consider

$$N|0\rangle = a^\dagger a(a^m|\psi\rangle) = 0 = a^\dagger(a^{m+1}|\psi\rangle) = 0. \tag{15}$$

where (Eq.(14)) has been used in the last step.
To summarize, we have the results that $|0\rangle$ is a normalized eigenvector of $N$ with zero as an eigenvalue and satisfies

$$a|0\rangle = 0. \tag{16}$$

[1] We use $(\phi, \chi)$, as well as Dirac notation $\langle\phi|\chi\rangle$, to denote scalar product of two vectors $\phi, \chi$.

## The eigenvalues of $N$ are all non negative integers

We will now show that $(a^\dagger)^r \equiv |\phi_r\rangle$ raises eigenvalue of $N$ by $r$ units. In other words $(a^\dagger)^r|0\rangle$ is an eigenvector of $N$ with eigenvalue $r$. This proof will be completed by the method of induction.

The give statement is obviously true for $m = 0$. Let us now assume the statement be true for $r = m$, *i.e.* we assume

$$N|\phi_m\rangle = m|\phi_m\rangle. \tag{17}$$

to be true and prove the statement for $r = m + 1$:

$$\begin{aligned} N|\phi_{m+1}\rangle &= N(a^\dagger)^{m+1}|0\rangle = Na^\dagger|\phi_m\rangle \\ &= (a^\dagger N + a^\dagger)|\phi_m\rangle = a^\dagger\,(m+1)|\phi_m\rangle \\ &= (m+1)(a^\dagger|\phi_m\rangle) = (m+1)|\phi_{m+1}\rangle. \end{aligned} \tag{18}$$

Therefore, $|\phi_{m+1}\rangle$ is an eigenvector of $N$ with eigenvalue $(m+1)$. To summarize, we have the result that the eigenvalues of $N$ are given by $0, 1, 2, \cdots, m, \cdots$. The corresponding normalised eigenvectors will be denoted by

$$|0\rangle, |1\rangle, |2\rangle, \cdots, |m\rangle, \cdots.$$

These are also eigenvectors of $\hat{H}$, the ket $|n\rangle$ corresponds to the eigenvalue $(n + \frac{1}{2})\hbar\omega$, because

$$\hat{H}|n\rangle = (N + \tfrac{1}{2})\hbar\omega\,|n\rangle = \hbar\omega\left(N|n\rangle + \tfrac{1}{2}|n\rangle\right) = (n + \tfrac{1}{2})\hbar\omega|n\rangle. \tag{19}$$

The third postulate of quantum mechanics tells us that the allowed energies coincide with the eigenvalues of the Hamiltonian. Hence the desired energy levels of of the harmonic oscillator are

$$\boxed{E_n = (n + \tfrac{1}{2})\hbar\omega.} \tag{20}$$

## Properties of eigenvectors

Let $|n\rangle$ be the normalised eigenvector of $N$ with eigenvalue $n$. Action of $a^\dagger$ on $|n\rangle$ gives a vector proportional to $|n+1\rangle$ and we write

$$a^\dagger|n\rangle = c_n|n+1\rangle. \tag{21}$$

The constant $c_n$ can be found by taking the inner product of vector in Eq.(21) with itself. This leads to

$$\langle n|aa^\dagger|n\rangle = |c_n|^2\langle n+1|n+1\rangle = |c_n|^2. \tag{22}$$

The left hand side of Eq.(22) is seen to be $(n + 1)$ by computing

$$\begin{aligned} aa^\dagger|n\rangle &= (a^\dagger a + 1)|n\rangle, \qquad (\because [a, a^\dagger] = 1) \\ &= (N+1)|n\rangle = (n+1)|n\rangle \qquad (23) \\ \therefore \quad \langle n|aa^\dagger|n\rangle &= n + 1 \qquad (24) \end{aligned}$$

which when used in Eq.(22) gives $(n + 1) = |c_n|^2$ and hence the result

$$\boxed{a^\dagger|n\rangle = \sqrt{(n+1)}|n+1\rangle}. \tag{25}$$

In a similar fashion (verify!) we get

$$\boxed{a|n\rangle = \sqrt{(n-1)}|n-1\rangle}. \tag{26}$$

Recursive use of Eq.(24) leads to the result that all states $|n\rangle$ can be obtained by applying powers of $a^\dagger$ on $|0\rangle$:

$$|n\rangle = \frac{1}{\sqrt{n}} a^\dagger |n-1\rangle = \frac{1}{\sqrt{n(n-1)}} (a^\dagger)^2 |n-2\rangle = \cdots = \frac{1}{\sqrt{n!}} (a^\dagger)^n |0\rangle. \tag{27}$$

## Questions

[1] Writing opertor order carfully, verify Eq.(6).

[2] Use method mathematical induction to show that

$$[N, a^r] = -rN, \qquad [N, (a^\dagger)^r] = rN. \tag{28}$$

Use the above results to show that $(a^\dagger)^r$ raises the eigenvalue of $N$ in step of $r$ and $a^r$ lowers the eigenvalue in step of $r$. (Eq.(12)).

[3] Check carefully if you have followed steps in Eq.(8).

[4] Prove that if eigenvalues of a hermitian operator are positive, its expectation value in every state is positive.

[5] Define eigenvalue and eigenvector of an operator carefully.

[6] Can zero be an eigenvalue? Can null vector be an eigenvector of an operator?

[7] One can always find a large $r$ so that Eq.(13) holeds. Is a corresponding statement true for $(a^\dagger)^r$ ? WHY??

[8] The lowest energy state of harmonic oscillator has non zero energy. This is related to uncertainty relation. The details are given in every text book and must be most frequently asked question in interviews. So if you do not know already this fact, learn from a text book of your choice.

[9] Make a flow chart for this derivation of energy levels of harmonic oscillator.

[10] Remember that $c_n$ in Eq.(22) is a complex number whose absolute value is determined to be $\sqrt{(n+1)}$ and the phase is undetermined. The phase has been taken to be zero and $c_n$ is fixed as real positive constant. Does this matter? WHY?

The issue of fixing phases will be discussed when it is important to address it, otherwise the phase will be taken to be zero.

[11] Find dual vectors $\langle 0|a$, $\langle 0|a^\dagger$, $\langle n|(a)^m$, $\langle n|(a^\dagger)^m$.

QM-CAPS-08001.tex; Aug 21, 2013