Two Qubit Gates

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1 Two Qubit Gates

Gates acting on two qubits $|x\rangle|y\rangle$ are known as two qubits.
Two qubit $|x\rangle|y\rangle$, more generally $|\psi_1\rangle|\psi_2\rangle$ are elements of $\mathcal{H}\otimes\mathcal{H}\equiv\mathcal{H}^{(2)}$

CNOT, or controlled NOT, gate is a two qubit gate. It action on $|x\rangle|y\rangle$ leaves the first qubit $|x\rangle$, called control qubit, unchanged and flips the second qubit $|y\rangle$ if the control bit is $|1\rangle$

$$\begin{array}{ll}\text{CNOT } |0\rangle|0\rangle = |0\rangle|0\rangle & \text{CNOT } |0\rangle|1\rangle = |0\rangle|1\rangle \\ \text{CNOT } |1\rangle|0\rangle = |1\rangle|1\rangle & \text{CNOT } |1\rangle|1\rangle = |1\rangle|0\rangle\end{array}$$

Here $x, y \to \{0 \,\text{or}\, 1\}$.

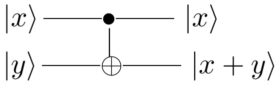
More generally, using the notation $|x\rangle|y\rangle \equiv |xy\rangle$, we have

$$\begin{aligned} & \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle \\ \xrightarrow{CNOT}\; & \alpha|00\rangle + \beta|01\rangle + \gamma|11\rangle + \delta|10\rangle \end{aligned}$$

1 Matrix representation

$$CNOT = \left[\begin{array}{c|c} I_2 & 0 \\ \hline 0 & \sigma_x \end{array}\right]$$

Digramatically



where $x, y = 0, 1$

CNOT gate

2 Exercise/Example

1. The figure shows a Hadamard gate on the first qubit followed by a CNOT gate on $|00\rangle$. It will give

$$|00\rangle \overset{H}{\rightarrowtail} \frac{|0\rangle + |1\rangle}{\sqrt{2}} \oplus |0\rangle \xrightarrow{CNOT} \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

   Exercise find action of the above network on $|01\rangle, |10\rangle, |11\rangle$.

2. Frequently one needs NOT operation on a single qubit

$$|0\rangle \overset{NOT}{\rightarrowtail} |1\rangle, \qquad |1\rangle \overset{NOT}{\rightarrowtail} |0\rangle \tag{1}$$

   This operation can be achieved by adding a control bit $|1\rangle$ and using CNOT gate as follows.

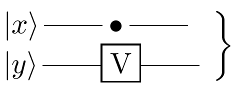
$$|1\rangle|0\rangle \overset{NOT}{\rightarrowtail} |1\rangle|1\rangle, \qquad |1\rangle|1\rangle \overset{NOT}{\rightarrowtail} |1\rangle|0\rangle \tag{2}$$

More 2 qubit gates

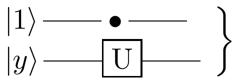
The action of Hadamard gate followed by a CNOT gate on two qubits is as follows.

$$\begin{aligned}
|00\rangle = |0\rangle|0\rangle &\overset{H}{\rightarrowtail} \frac{|0\rangle + |1\rangle}{\sqrt{2}}|0\rangle \overset{CNOT}{\rightarrowtail} \frac{|00\rangle + |11\rangle}{\sqrt{2}} \\
|10\rangle = |1\rangle|0\rangle &\overset{H}{\rightarrowtail} \frac{|0\rangle - |1\rangle}{\sqrt{2}}|0\rangle \overset{CNOT}{\rightarrowtail} \frac{|00\rangle - |11\rangle}{\sqrt{2}} \\
|x_1x_2\rangle = |x_1\rangle|x_2\rangle &\longrightarrow ?? \qquad \text{where } x_1, x_2 \in \{0, 1\}
\end{aligned}$$

2 qubit controlled phase gate



$|0\rangle|y\rangle$ does nothing to $|y\rangle$



$|1\rangle(U|y\rangle)$ applies unitary operator u only $|y\rangle$

In a controlled gate the control bit(s) do not change

$$|x\rangle|1\rangle \overset{CNOT}{\rightarrowtail} |x\rangle|x \oplus 1\rangle$$

so $|x\rangle|0\rangle \rightarrowtail |x\rangle|x\rangle$.

CNOT gate has representation

$$\text{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

in computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$

CV gate is the gate given by the matrix

$$V = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

CV gate is a controlled phase gate where the phase matrix is V. Note that V is unitary and $V^4 = I$.

Example: A CNOT gate can be built from H and CV gates

Check for missing figures