Single Qubit Quantum Gates

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1 Single Qubit Quantum Gates

1 Hadamard gate

The Hadamard gate can be represented by a matrix

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

in computational bases the states $|0\rangle$ and $|1\rangle$ transform as

$$H|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \qquad H|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

<u>**Example**</u> For $|\pm\rangle = \frac{|0\rangle \pm |1\rangle}{\sqrt{2}}$

$$H|+\rangle = |0\rangle \ , \quad H|-\rangle = |1\rangle$$

The action of Hadamard gate is unitary operation.

The action of Hadamard gate is diagrammatically represented as

$$|0\rangle \;—\!\!—\boxed{\mathrm{H}}—\!\!—\; \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$|1\rangle \;—\!\!—\boxed{\mathrm{H}}—\!\!—\; \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

$$\text{or} \quad |x\rangle \;—\!\!—\boxed{\mathrm{H}}—\!\!—\; \frac{1}{\sqrt{2}}\{(-1)^x \ |x\rangle + |1-x\rangle\}$$

where $|x\rangle$ can be $|0\rangle$ or $|1\rangle$.
Ref. Ekert, A. "Basic Concepts in Quantum Computation", arxiv-quant-pn/0011013v1, Nov. 2000.

2 Rotation about X, Y, Z axes

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$$\begin{aligned} X, Y, Z \longrightarrow & \text{ Pauli Matrices} \\ & X = \sigma_x \qquad Y = \sigma_y \qquad Z = \sigma_z \\ & = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned}$$

Operation of rotation about X, Y, Z axes can be built in terms of Pauli matrices

$$R_x(\theta) = \exp\left(-i\alpha\frac{X}{2}\right) = \begin{pmatrix} \cos(\alpha/2) & -i\sin(\alpha/2) \\ -i\sin(\alpha/2) & \cos(\alpha/2) \end{pmatrix}$$

performs a rotation about X axis.

$$R_y(\theta) = \exp\left(-i\theta\frac{Y}{2}\right) \quad \text{and} \quad R_z(\theta) = \exp\left(-i\theta\frac{Z}{2}\right)$$

rotate the state of a qubit about Y and Z axes. The matrix $U(\phi) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix}$ changes phase of $|1\rangle$ and leaves $|0\rangle$ unchanged

$$U(\phi)|0\rangle = |0\rangle \qquad U(\phi)|1\rangle = e^{i\phi}|1\rangle\,.$$

An arbitrary rotation about $\hat{n} = (n_1, n_2, n_3)$ by angle α can be written in terms of R_x, R_y and R_z

Question 1 *Write an arbitrary rotation*

$$U(\hat{n}, \alpha) = \exp\left[-i\alpha\big(n_1 X + n_2 Y + n_3 Z\big)\right]$$

in terms of rotations R_x, R_y, R_z about the coordinate axes.

The phase gate ϕ changes the phase of a qubit.

The usefulness of R_x, R_y, R_z and ph lies in the fact that an arbitrary unitary operator V with determinant $\det V = 1$ has a representation in terms of rotations about the coordinate axes

$$V \equiv R_x(\alpha) R_y(\beta) R_z(\gamma).$$

An arbitrary unitary matrix can be written as a product of rotations about the coordinate axes and a phase transformation

$$U = R_z(\alpha)\ R_y(\beta)\ R_z(\gamma)\ ph(\delta).$$

Transformation from $|0\rangle$ to an arbitrary state

A general state, θ, ϕ on Bloch sphere can be obtained from $|0\rangle$ by a sequence of operations

$$|0\rangle \;\text{——}\boxed{\mathrm{H}}\text{——}\; \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$\frac{|0\rangle + |1\rangle}{\sqrt{2}} \;\text{——}\boxed{2\theta}\text{——}\; \frac{|0\rangle + e^{2i\theta}|1\rangle}{\sqrt{2}}$$

$$\frac{e^{-i\theta}|0\rangle + e^{i\theta}|1\rangle}{\sqrt{2}} \;\text{——}\boxed{\mathrm{H}}\text{——}\; \frac{1}{\sqrt{2}}e^{-i\theta}\left\{|0\rangle + |1\rangle\right\} + \frac{1}{\sqrt{2}}e^{i\theta}\left\{|0\rangle - |1\rangle\right\}$$

$$= \cos\theta|0\rangle - i\sin\theta|1\rangle$$

$$\cos\theta|0\rangle - i\sin\theta|1\rangle \;\text{——}\boxed{\pi/2 + \phi}\text{——}\; \cos\theta|0\rangle + \sin\theta e^{i\phi}|1\rangle$$

Thus a single qubit $|0\rangle$ can be transformed into a general state $|\theta, \phi\rangle$ by using Hadamard and phase gates

$$|0\rangle \;\text{——}\boxed{\mathrm{H}}\overset{2\theta}{\text{——}}\boxed{\mathrm{H}}\overset{\pi/2+\phi}{\text{——}}\; \cos\theta|0\rangle + \sin\theta e^{i\theta}|0\rangle$$

In general an n qubit state $|\psi_1\rangle|\psi_2\rangle \cdot |\psi_n\rangle$, can be generated from $|0\rangle|0\rangle \cdots |0\rangle$ using Hadamard gates and phase gates.

Question 2 *Verify the following identities:*

$$ph(\theta)|0\rangle = e^{i\phi}|0\rangle \quad ph(\phi)|1\rangle = e^{i\phi}|1\rangle.$$