Notes for Quantum Computation and Quantum Information

Density Matrix of Mixed State

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Example Given mixed a state $|\psi\rangle$, the corresponding density matrix is uniquely determined as $\rho = |\psi\rangle\langle\psi|$. However the same is not true for a mixed state. We shall demonstrate that same averages can be obtained by two different density matrices.

Consider a density matrix for spin half system given by

$$\rho = \frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1| \tag{1}$$

Let $|a\rangle, |b\rangle$ be two orthogonal vectors defined as follows.

$$|a\rangle = \sqrt{\frac{3}{4}}|0\rangle + \sqrt{\frac{1}{4}}|1\rangle \tag{2}$$

$$|b\rangle = \sqrt{\frac{3}{4}}|0\rangle + \sqrt{\frac{1}{4}}|1\rangle \tag{3}$$

So that

$$|0\rangle = \frac{1}{\sqrt{3}}(|a\rangle + |b\rangle), \qquad |1\rangle = (|a\rangle - |b\rangle) \tag{4}$$

The given density matrix takes the form

$$\begin{aligned} \rho &= \frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1| \\ &= \frac{3}{4}\Big(\frac{1}{3}(|a\rangle + |b\rangle)(\langle a| + \langle b|)\Big) + \frac{1}{4}\Big((|a\rangle - |b\rangle)(\langle a| - \langle b|)\Big) \qquad (5) \\ &= \frac{1}{2}|a\rangle\langle a| + \frac{1}{2}|b\rangle\langle b| \qquad (6) \end{aligned}$$

Thus same averages can be obtained from two different ensembles. In general if we are given a density matrix

$$\rho = \sum_n p_n|\psi_n\rangle\langle\psi_n| \tag{7}$$

where $\{|\psi_n\rangle\}$ are a set of orthogonal vectors. Let U be a unitary matrix. Define $|\phi_n\rangle$ by

$$|\phi_n\rangle = \sum_n U_{nm}|\psi_m\rangle. \tag{8}$$

Then the set of vectors $\{|\phi_n\rangle\}$ is an orthogonal set. The density matrix $\tilde{\rho}$ defined as

$$\tilde{\rho} = \sum_n p_n|\phi_n\rangle\langle\phi_n| \tag{9}$$

gives the same average for every observable X. To prove this statement let us define the matrix element

$$(X)_{mn} = \langle\psi_m|\widehat{X}|\psi_n\rangle \tag{10}$$

Then the average of $\widehat{X}$ is given by

$$\begin{aligned}
\widehat{X}_\rho &= \mathrm{tr}\rho\widehat{X} && (11)\\
&= \sum_{mn}\rho_{nm}(X)_{mn} && (12)\\
&= \sum_{mn}\langle n|\rho|m\rangle\cdot\langle m|\widehat{X}|n\rangle && (13)\\
&= \sum_{mn}\sum_j p_j\langle n|\psi_j\rangle\langle\psi_j|m\rangle\cdot\langle m|\widehat{X}|n\rangle && (14)\\
&= \sum_j p_j\langle\psi_j|\widehat{X}|\psi_j\rangle && (15)\\
&= \sum_j p_j\mathrm{tr}\big(\widehat{X}|\psi_j\rangle\langle\psi_j|\big). && (16)
\end{aligned}$$

Since U is a unitary transformation

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Bloch representation A general density matrix for mixed state of a two level system is

$$\rho = \frac{1}{2}\Big(I + \vec{r}\cdot\vec{\sigma}\Big)$$

Problem Show thatthe above density matrix represents a pure state if and only if vector $\vec{r}$ is a unit vector, $|\vec{r}| = 1$.

Problem Give explicit expressions of two different density matrices for a two level system which will give rise to the same average values.