Bloch Sphere

A. K. Kapoor

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1 Bloch Sphere:

A qubit may be thought of a spin $\frac{1}{2}$ state vector. If spin projection along direction (θ, ϕ) is $+\frac{1}{2}$ the corresponding state vector must be eigen vector of $(\hat{n} \cdot \vec{\sigma})$ with eigen value 1.

$$(\hat{n} \cdot \vec{\sigma}) \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{1}$$

Writing

$$\hat{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), \tag{2}$$

the normalized eigenvector can be written in the form

$$\chi = \begin{pmatrix} \cos(\theta/2)e^{-i\phi} \\ \sin(\theta/2) \end{pmatrix} \tag{3}$$

comparing with $\binom{\alpha}{\beta}$ we see that

$$|\alpha| = \cos\frac{\theta}{2} \qquad |\beta| = \sin\frac{\theta}{2}, \qquad 0 \le \theta \le n \tag{4}$$

$$e^{-i\phi} = \frac{\alpha}{|\alpha|}. \qquad 0 \le \phi < 2\pi \tag{5}$$

Thus qubits, apart from an over all phase, are in one to one correspondence with points (θ, ϕ) on a unit sphere. Representation of qubit by a point on sphere is known as the Bloch sphere representation.

Representation of pure and mixed states

The density operator corresponding to the pure state (3) is

$$\rho = \frac{1}{2}(I + \hat{n} \cdot \vec{\sigma}) \tag{6}$$

with $|\hat{n}|$ given by (2).

For $|\vec{m}| < 1$, the vector $\vec{m}$ represents a point inside the Bloch sphere and the density operator

$$\rho = \frac{1}{2}(I + \vec{m} \cdot \vec{\sigma}), \qquad |\vec{m}| < 1 \tag{7}$$

represents a mixed state since $\rho^2 \neq \rho$.

Thus the pure states of qubit are in a one to one correspondence with points on the sphere (pure states). The points inside the Bloch sphere correspond to possible mixed states of a qubit.

Question for You: Compute eigenvectors of $\hat{n} \cdot \vec{\sigma}$ with eigen values ± 1. Verify that a normalized eigenvector with eigen values ± 1 can be cast in the form (3).

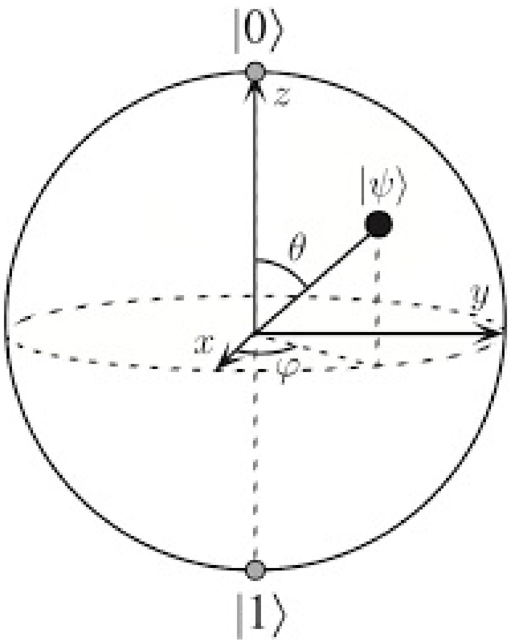

Fig. **1**