LaTeX-Any Document

Positive Operators

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1 Positive Operators

Definition 1 *An operator P, in a complex inner product space, is called* **positive operator** *if*

$$\langle f|A|f\rangle \geq 0\,. \tag{1}$$

holds for all vectors $|f\rangle$ in the vector space.

2 Properties of positive operators

(a) The eigenvalues of a positive operator are positive.

Proof If P is a positive operator and α is an eigenvalue with eigenvector $|\alpha\rangle$

$$P|\alpha\rangle = \alpha|\alpha\rangle \Rightarrow \frac{\langle\alpha|P|\alpha\rangle}{\langle\alpha|\alpha\rangle} = \alpha \tag{2}$$

Therefore $\alpha \geq 0$. Thus the eigenvalues of a positive operator are non negative.

(b) A positive operator in an inner product space is hermitian.

Proof We shall use the fact that $\langle f|X|f\rangle = 0$ for all vectors implies that the operator $X = 0$

$$\langle f|X|f\rangle = 0 \Rightarrow X = 0\,. \tag{3}$$

For a positive operator $\langle f|P^\dagger|f\rangle = \langle f|P|f\rangle^* = \langle f|P|f\rangle$. The last step follows from positivity property of operator P. Therefore

$$\langle f|(P - P^\dagger)|f\rangle = \langle f|P^\dagger|f\rangle - \langle f|P|f\rangle = 0 \tag{4}$$

holds for every vector in $|f\rangle$ in the vector space. This implies that $P - P^\dagger = 0$ or P is a hermitian operator.

3 Proof of (3):

For completeness we write out proof of (3). This result is a consequence of polarization like identity

$$\begin{aligned} 4i\langle f|X|g\rangle &= \langle (f+g)|X|(f+g)\rangle - \langle (f-g)|X|(f-g)\rangle \\ &\quad + i\langle (f+ig)|X|(f+ig)\rangle - i\langle (f-ig)|X|(f-ig)\rangle \end{aligned} \tag{5}$$

If $\langle f|X|f\rangle$ vanishes for all f, the r.h.s. of (5) is zero giving $\langle f|X|g\rangle = 0$ for all vectors $|f\rangle$ and $|g\rangle$. This in turn implies that the operator X itself is zero.

4 Example:

An operator A having positive eigenvalues need not be a positive operator. To see this, consider $A' = SAS^{-1}$ has positive eigenvalues, for every invertible operator S. If A is hermitian, A' need not be hermitian. Since positive operators are necessarily hermitian, we conclude that A need not be positive operator.

Question for you: In a finite dimensional complex vector space, give an example of an operator X and vector $|f\rangle$ such that the eigenvalues of X are positive but $\langle f|X|f\rangle$ is negative.