# Notes for Lectures on Quantum Information and Quantum Computation

## Trace and Partial Trace

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### §1 Some notation

We will work with Dirac bra ket notation.The elements of a vector space are denoted by kets, for example $|\psi\rangle$. The elements of the dual vector space will be denoted by bras, for example $\langle\psi|$.

An object written as 'outer product', $|\phi_1\rangle\langle\phi_2| \equiv \hat{X}$, is a linear operator in the vector space. The action of operator $\hat{X}$ on an arbitrary vector $|\psi\rangle$ is given by

$$\hat{X}|\psi\rangle = \langle\phi_2|\psi\rangle\,|\phi_1\rangle. \tag{1}$$

If $\{|n\rangle\}$ is an orthonormal basis in the vector space, then it satisfies the following mathematical properties.

**Orthonormal property** $$\langle m|n\rangle = \delta_{mn} \tag{2}$$

**Completeness property** $$\sum_n |n\rangle\langle n| = \hat{I}, \tag{3}$$

where $\hat{I}$ is identity operator.

Using an orthonormal basis $\{|j\rangle\}$, every linear operator, $\hat{X}$, in a vector space can be represented by a matrix,$\underline{X}$, with its elements $(\underline{X})_{jk}$ given by

$$(\underline{X})_{jk} = \langle j|\underline{\hat{X}}|k\rangle\ , \qquad j,k = 1,\cdots n \tag{4}$$

where $\{|j\rangle, j = 1,\cdots n\}$ is an orthonormal basis. The matrix representing the operator depends on choice of the basis Eq.(4) is equivalent to operator relation

$$\hat{X} = \sum_{j,k=1}^{N} X_{jk}|j\rangle\langle k|. \tag{5}$$

A useful quantity, trace of an operator $\hat{X}$, is defined to be just the sum of diagonal elements of the matrix $\underline{X}$:

$$\begin{aligned} tr(X) &= \sum_{j=1}^{N} (\underline{X})_{jj} \\ &= \sum_{j=1} \langle j|\hat{X}|j\rangle \end{aligned} \tag{6}$$

From (4) and (5) it is easy to verify that the trace can be cast in the form

$$tr(X) = \sum_{n=1}^{N} tr(X|n\rangle\langle n|) \tag{7}$$

This relation becomes transparent on recalling that for an orthonormal basis we have the completeness relation $\sum_{n=1}^{N} |n\rangle\langle n| = I_n$

**Partial Trace:**

Let $\mathcal{H}_A, \mathcal{H}_B$ be two Hilbert spaces with orthonormal bases $\{|\nu A\rangle | \nu = 1 \cdots N\}$ and $\{|aB\rangle | a = 1 \cdots M\}$ respectively. Then the tensor product Hilbert space has a basis

$$\{|\nu A\rangle \otimes |aB\rangle \equiv |\nu A; aB\rangle|, \qquad \nu = 1 \cdots N, a = 1 \cdots M\}\,. \tag{8}$$

This basis will be orthonormal basis and can be used to construct a representation of vectors and operators in $\mathcal{H}_A \otimes \mathcal{H}_B$.
An operator $\widehat{\mathscr{X}}$ on the tensor product space will then be represented by a matrix $\underline{\mathscr{X}}$ with elements

$$(\underline{\widehat{\mathscr{X}}})_{\nu a, \lambda b} = \langle \nu A, aB|\widehat{\mathscr{X}}|\lambda A, bB\rangle \tag{9}$$

The trace of operator $\widehat{\mathscr{X}}$ is as usual defined to be

$$Tr(\widehat{\mathscr{X}}) = \sum_{a=1}^{M} \sum_{\nu=1}^{N} \langle \nu A, aB|\widehat{\mathscr{X}}|\nu A, Ba\rangle \tag{10}$$

If we sum over only one of the two indices $a$ (or $\nu$) we get partial trace of operator $\widehat{\mathscr{X}}$.

$$\left( \widehat{\mathscr{X}}\Big|_{trB} \right)_{\nu\lambda} = \sum_{a=}^{M} \langle \nu A, aB|\widehat{\mathscr{X}}|\lambda A, aB\rangle \tag{11}$$

the above expression defines an operator $\widehat{\mathscr{X}_A}$ in $\mathcal{H}_A$ with matrix elements given by r.h.s. of (11). $\widehat{\mathscr{X}_A}$ will be called particle trace of $\widehat{\mathscr{X}}$ over $B$.

Similarly $\widehat{\mathscr{X}_B}$, the result of partial of $\widehat{\mathscr{X}}$ over $A$ will be an operator in $\mathcal{H}_B$ and is defined in a smilar manner.

$$\widehat{\mathscr{X}_A} = \sum_{\nu\lambda} \left( \widehat{\mathscr{X}}\Big|_{trB} \right)_{\nu\lambda} |\nu\rangle\langle\lambda| \qquad (12)$$

$$\widehat{\mathscr{X}_B} = \sum_{a,b} \left( \widehat{\mathscr{X}}\Big|_{trA} \right)_{ab} |a\rangle\langle b| \qquad (13)$$

**Remarks:**

The trace of an operator, written as $P = |\phi\rangle\langle\psi|$, is easily computed. Let $\{|n\rangle\}$ be an orthonormal basis. Then

$$\begin{aligned} Tr(P) &= \sum_n \langle n|P|n\rangle = \sum_n \langle n|\phi\rangle\langle\psi|n\rangle \\ &= \sum_n \langle\psi|n\rangle\langle n|\phi\rangle \\ \therefore\ Tr(|\phi\rangle\langle\psi|) &= \langle\psi|\phi\rangle \end{aligned}$$

In general, for an operator, $P$, having a form

$$\begin{aligned} P &= \sum_{i,j} p_{ij} |\psi_i\rangle\langle\phi_j| \\ Tr(P) &= \sum_{i,j} p_{ij} \langle\phi_j|\psi_i\rangle \,. \end{aligned}$$