Notes for Lectures in Quantum Mechanics *

Qubits

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1 Two level system as a qubit

One way to define qubits is to use quantum states of a spin half system. The spin "down" and spin "up" states of a spin half system can be represented as $\longrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. We represent these special states as $|0\rangle, |1\rangle$. A spin half system can also be thought of as a two level quantum system.

A general state of a two level system is a linear superposition

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \longrightarrow \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

We shall use x and y, for 0 or 1 exclusively. Corresponding qubits will be represented as $|x\rangle, |y\rangle$ respectively.

$$|x\rangle \longrightarrow |0\rangle \text{ or } |1\rangle \text{ (but not a superposition!)}$$

A general superposition will be denoted by a Greek letter such as $|\chi\rangle$. Thus we shall write (special) qubits $|0\rangle, |1\rangle$ as $|x\rangle$ with x taking values $x \in \{0, 1\}$.

*qbit; Updated:Nov 15, 2021; Ver 0.x

With spin along a unit vector $\hat{n}$, the corresponding state $|x\rangle$ is determined by

$$(\hat{n}\cdot\hat{\sigma})|\chi\rangle = |\chi\rangle$$
$$\text{or} \qquad (\hat{n}\cdot\hat{\sigma})\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}\alpha\\ \beta\end{pmatrix} \tag{1}$$

In this notation, if we measure spin component $(\hat{n}\cdot\hat{s})$ the outcome will be $+\hbar/2$.

Note that a unit vector $\hat{n}$ represents a direction in 3 dimensions. The direction can also be represented by polar angles (θ,ϕ). The correspondence of the polar angles with a unit vectors is given by

$$\hat{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \tag{2}$$

where the polar angles have the range $0 \le \theta < \pi, 0 \le \phi < 2\pi$. Thus, if a pure state $\binom{a}{b}$ is given, we can always find $\hat{n}$ (or θ,ϕ) by making use of (1)

2 Bloch Sphere

We have seen that a pure state of the two level system is in correspondence with θ,ϕ. The polar angles determine a point on the unit sphere. The set of all points on, and inside unit sphere is called Bloch Sphere
Pure states are represented by vectors in vector space corresponding to $\hat{n}\cdot\hat{n} = 1$. These point lie on the surface of the Bloch sphere.

3 Mixed States

To describe a mixed state of a spin half system, we need density matrix. A general density matrix is

$$\rho = \frac{1}{2}(\vec{I} + \vec{n}\cdot\vec{\sigma}), \qquad tr\rho = 1$$

and $tr\ \rho^2$ is

$$tr\rho^2 = \frac{1}{4}(2 + 2\vec{n}\cdot\vec{n}) \le 1 \tag{3}$$

Therefore, $tr(\rho^2) \le tr\rho$ for vectors with length less than 1. These vectors have $\vec{n}\cdot\vec{n} < 1$ and correspond to the points inside Bloch sphere. Thus a point inside the Bloch sphere represents a mixed state. When $\vec{n} \cdot \vec{n} = 1$ $\rho^2 = \rho$ and this case corresponds to pure state. A pure state thus corresponds to a point on the surface of the Bloch sphere.