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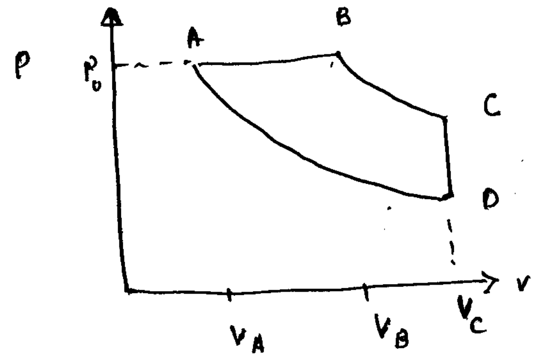

Note $V_B = \frac{RT_B}{P_0}$; $V_A = \frac{RT_A}{P_0}$

$(T_B > T_A)$

$P_C V_C = RT_C$

$P_D V_C = RT_D$

$(T_C > T_D)$

Heat absorbed by the system $= C_p(T_B - T_A)$

Heat given out by the system $= C_v(T_C - T_D)$

$$\eta = 1 - \frac{C_v(T_C - T_D)}{C_p(T_B - T_A)}$$

BC is adiabatic : $T_B V_B^{\gamma-1} = T_C V_C^{\gamma-1}$ $(V_C = V_D)$

AD is " : $T_A V_A^{\gamma-1} = T_D V_C^{\gamma-1}$

$$T_C - T_D = \frac{T_B V_B^{\gamma-1}}{V_C^{\gamma-1}} - \frac{T_A V_A^{\gamma-1}}{V_C^{\gamma-1}} = \frac{1}{V_C^{\gamma-1}}\left[T_B V_B^{\gamma-1} - T_A V_A^{\gamma-1}\right]$$

$$= \frac{1}{V_C^{\gamma-1}}\left[\frac{P_0 V_B^{\gamma}}{R} - \frac{P_0 V_A^{\gamma}}{R}\right]$$

$$= \frac{P_0}{R}\frac{1}{V_C^{\gamma-1}}\left[V_B^{\gamma} - V_A^{\gamma}\right]$$

$$T_B - T_A = \frac{P_0}{R}\left[V_B - V_A\right]$$

$$\eta = 1 - \frac{1}{\gamma}\frac{V_C\left[\left(\frac{V_B}{V_C}\right)^{\gamma} - \left(\frac{V_A}{V_C}\right)^{\gamma}\right]}{V_B - V_A}$$

$$= 1 - \frac{1}{\gamma}\frac{\left[\left(\frac{V_B}{V_C}\right)^{\gamma} - \left(\frac{V_A}{V_C}\right)^{\gamma}\right]}{\left(\frac{V_B}{V_C}\right) - \left(\frac{V_A}{V_C}\right)}$$

Remark : 1. Efficiency depends on the equation of state

(14) R.H.S $\left.\frac{\partial x}{\partial c}\right|_d \left.\frac{\partial y}{\partial d}\right)_c - \left.\frac{\partial y}{\partial c}\right)_d \left.\frac{\partial x}{\partial d}\right)_c$

$$= \left[\overset{1}{\left.\frac{\partial x}{\partial a}\right)_b \left.\frac{\partial a}{\partial c}\right)_d} + \overset{2}{\left.\frac{\partial x}{\partial b}\right)_a \left.\frac{\partial b}{\partial c}\right)_d} \right] \left[\overset{1'}{\left.\frac{\partial y}{\partial a}\right)_b \left.\frac{\partial a}{\partial d}\right)_c} + \overset{2'}{\left.\frac{\partial y}{\partial b}\right)_a \left.\frac{\partial b}{\partial d}\right)_c} \right]$$

$$- \left[\overset{A}{\left.\frac{\partial y}{\partial a}\right)_b \left.\frac{\partial a}{\partial c}\right)_d} + \overset{B}{\left.\frac{\partial y}{\partial b}\right)_a \left.\frac{\partial b}{\partial c}\right)_d} \right] \left[\overset{A'}{\left.\frac{\partial x}{\partial a}\right)_b \left.\frac{\partial a}{\partial d}\right)_c} + \overset{B'}{\left.\frac{\partial x}{\partial b}\right)_a \left.\frac{\partial b}{\partial d}\right)_c} \right]$$

$$1\,1' + A\,A' = 0 \quad ; \quad 2\,2' + B\,B' = 0$$

Rest

$$\text{R.H.S} = \underset{\text{I}}{\left.\frac{\partial x}{\partial a}\right)_b \left.\frac{\partial a}{\partial c}\right)_d \left.\frac{\partial y}{\partial b}\right)_a \left.\frac{\partial b}{\partial d}\right)_c} + \underset{\text{II}}{\left.\frac{\partial x}{\partial b}\right)_a \left.\frac{\partial b}{\partial c}\right)_d \left.\frac{\partial y}{\partial a}\right)_b \left.\frac{\partial a}{\partial d}\right)_c}$$

$$- \underset{\text{III}}{\left.\frac{\partial y}{\partial a}\right)_b \left.\frac{\partial a}{\partial c}\right)_d \left.\frac{\partial x}{\partial b}\right)_a \left.\frac{\partial b}{\partial d}\right)_c} - \underset{\text{IV}}{\left.\frac{\partial y}{\partial b}\right)_a \left.\frac{\partial b}{\partial c}\right)_d \left.\frac{\partial x}{\partial a}\right)_b \left.\frac{\partial a}{\partial d}\right)_c}$$

$$= \overset{\text{I + IV}}{\left.\frac{\partial x}{\partial a}\right)_b \left.\frac{\partial y}{\partial b}\right)_a} \left[\left.\frac{\partial a}{\partial c}\right)_d \left.\frac{\partial b}{\partial d}\right)_c - \left.\frac{\partial b}{\partial c}\right)_d \left.\frac{\partial a}{\partial d}\right)_c \right]$$

$$- \overset{(\text{III + IV})}{\left.\frac{\partial y}{\partial a}\right)_b \left.\frac{\partial x}{\partial b}\right)_a} \left[\left.\frac{\partial a}{\partial c}\right)_d \left.\frac{\partial b}{\partial d}\right)_c - \left.\frac{\partial b}{\partial c}\right)_d \left.\frac{\partial a}{\partial d}\right)_c \right]$$

$$= \left[\left.\frac{\partial x}{\partial a}\right)_b \left.\frac{\partial y}{\partial b}\right)_a - \left.\frac{\partial y}{\partial a}\right)_b \left.\frac{\partial x}{\partial b}\right)_a \right] \left[\left.\frac{\partial a}{\partial c}\right)_d \left.\frac{\partial b}{\partial d}\right)_c - \left.\frac{\partial b}{\partial c}\right)_d \left.\frac{\partial a}{\partial d}\right)_c \right]$$

$$= \frac{\partial(x, y)}{\partial(a, b)} \times \frac{\partial(a, b)}{\partial(c, d)} = \text{L.H.S.}$$

Remark : Can be generalised to higher dimensions.

15.

$$\left.\frac{\partial U}{\partial P}\right)_T = \begin{vmatrix} \left.\frac{\partial U}{\partial P}\right)_T & \left.\frac{\partial T}{\partial P}\right)_T \\ \left.\frac{\partial U}{\partial T}\right)_P & \left.\frac{\partial T}{\partial T}\right)_T \end{vmatrix} = \frac{\partial(U,T)}{\partial(P,T)}$$

$$= \frac{\partial(U,T)}{\partial(V,T)} \frac{\partial(V,T)}{\partial(P,T)} = \left.\frac{\partial U}{\partial V}\right)_T \times \left.\frac{\partial V}{\partial P}\right)_T$$

$= 0$ unless $\left.\frac{\partial V}{\partial P}\right)_T \to \infty$ or $\left.\frac{\partial P}{\partial V}\right|_T \to 0.$

(16)

$$H = U + PV \qquad dH = dU + P\,dV + V\,dP$$

$$T\,dS = \left.\frac{\partial U}{\partial T}\right)_H dT + \left.\frac{\partial U}{\partial H}\right)_T dH + p\left.\frac{\partial V}{\partial T}\right)_H dT + p\left.\frac{\partial V}{\partial H}\right)_T dH$$

$$= \left(\left.\frac{\partial U}{\partial T}\right)_H + p\left.\frac{\partial V}{\partial T}\right)_H\right) dT + \left[\frac{\partial U}{\partial H} dH + p\left.\frac{\partial V}{\partial H}\right)_T\right] dH$$

$$C_H = T\left.\frac{\partial S}{\partial T}\right)_H = \left.\frac{\partial U}{\partial T}\right)_H + p\left.\frac{\partial V}{\partial T}\right)_H$$

$$dH = T\,dS + V\,dp = dU + d(PV)$$

$$\left.\frac{\partial U}{\partial T}\right)_H = -\frac{d}{dT}(PV)\Big|_H$$

$$C_H = -\frac{d}{dT}(PV)\Big|_H + P\left.\frac{\partial V}{\partial T}\right)_H = -V\left.\frac{dP}{dT}\right)_H$$

$$\left.\frac{\partial p}{\partial T}\right)_H \left.\frac{\partial H}{\partial p}\right)_T \left.\frac{\partial T}{\partial H}\right)_p = -1$$

$$\left.\frac{\partial p}{\partial T}\right)_H = -\frac{\left.\frac{\partial H}{\partial T}\right)_p}{\left.\frac{\partial H}{\partial p}\right)_T} = -\frac{T\left.\frac{\partial S}{\partial T}\right)_p}{T\left.\frac{\partial S}{\partial p}\right)_T + V}$$

$$\begin{matrix} S & P \\ V & T \end{matrix}$$

$$= \frac{-C_p}{-T\left.\frac{\partial V}{\partial T}\right)_p + V} \qquad \left.\frac{\partial S}{\partial p}\right)_T = -\left.\frac{\partial V}{\partial T}\right|_p$$

$$C_H = \frac{C_p V}{V - T\left.\frac{\partial V}{\partial T}\right)_p} = \frac{C_p}{1 - \frac{T}{V}\left.\frac{\partial V}{\partial T}\right)_p}$$

17. $$P(n) = {}^{N}C_n \left(\frac{v}{V}\right)^n \left(1 - \frac{v}{V}\right)^{N-n}$$

$$= \frac{N!}{n!\,(N-n)!} \cdot \left(\frac{v}{V}\right)^n \left(1 - \frac{v}{V}\right)^{N-n} \qquad \frac{v}{V} = x$$

$$\ln P(n) = N \ln N - n \ln n - (N-n) \ln (N-n)$$
$$\underbrace{+ \frac{1}{2}\left[\ln (2\pi N) - \ln (2\pi n) - \ln 2\pi (N-n)\right]}_{\text{Ignore}}$$
$$+ n \ln x + (N-n) \ln (1-x)$$

$$\frac{\partial \ln P(n)}{\partial n} = -1 - \ln n + 1 + \ln (N-n) + \ln x - \ln (1-x) = 0$$

$$\ln \frac{N-n}{n} = \ln \frac{1-x}{x} \mp \underbrace{\frac{1}{2}\left[\frac{1}{n} + \frac{1}{N-n}\right]}_{\text{negligible contribution}}$$

$$Nx - nx = n - xn$$

$$\frac{N-n}{n} = \frac{1-x}{x}$$

$$\text{or} \quad x = \frac{n}{N} = \frac{v}{V}.$$

18. $$\frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2 < U$$

Consider $$\frac{p^2}{2mU} + \frac{x^2}{(2U/m\omega^2)} = 1 \qquad a^2 = 2mU, \quad b^2 = \frac{2U}{m\omega^2}.$$

$$\frac{p^2}{a^2} + \frac{x^2}{b^2} = 1$$

$$\text{Area} = \pi a b = \pi \sqrt{2mU} \sqrt{\frac{2U}{m\omega^2}} = \frac{2\pi}{\omega} U$$

$$\text{Area between } U \text{ \& } U + \Delta U = \frac{2\pi}{\omega} \Delta U$$

Note in the Q. case this becomes for the number of state $\quad \frac{2\pi}{\omega h} \Delta U = \frac{\Delta U}{\hbar\omega}$

$$= \underline{\underline{\Delta n}}.$$

21.

$$T\,dS = dU + p\,dV = dU + f(V)\,T\,dV$$

$$T\left[dS - f(V)\,dV\right] = dU$$

$$\left.\frac{\partial U}{\partial V}\right)_T = T\left.\frac{\partial S}{\partial V}\right)_T - T f(V) = T\left.\frac{\partial p}{\partial T}\right)_V - T f(V) = T f(V) - T f(V) = 0$$

20.

$$N! = \int_0^\infty e^{-x} x^N\,dx = \int_0^\infty e^{-x + N\ln x}\,dx$$

$$N g(x) = x - N\ln x \qquad g(x) = \frac{x}{N} - \ln x$$

Max at $\frac{1}{N} - \frac{1}{x} = 0$ or $x = N$. Let $x = N + \alpha$.

$$g(N+\alpha) = \frac{N+\alpha}{N} - \ln N - \ln\left(1 + \frac{\alpha}{N}\right)$$

$$= 1 + \frac{\alpha}{N} - \ln N - \frac{\alpha}{N} + \alpha^2/2N^2$$

$$N! = \int_{-N}^{\infty} e^{-N\left[1 - \ln N + \alpha^2/2N^2\right]}\,d\alpha$$

$$\simeq \int_{-\infty}^{\infty} e^{N\ln N - N}\, e^{-\alpha^2/2N^2}\,d\alpha$$

$$= e^{N\ln N - N}\int_{-\infty}^{\infty} e^{-\alpha^2/2N^2}\,d\alpha = e^{N\ln N - N}\sqrt{2\pi N}$$

$$\ln N! = N\ln N - N + \ln\sqrt{2\pi N}$$

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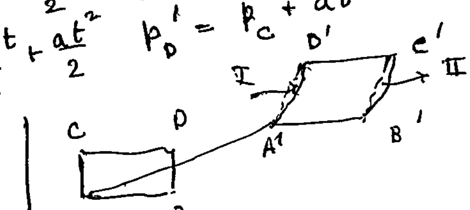
$$\left.\frac{\partial H}{\partial p}\right)_q = \frac{dq}{dt} = \frac{p}{m}. \qquad -\frac{\partial H}{\partial q} = \frac{dp}{dt} = ma \Rightarrow \frac{dq}{dt} = a$$

or $q = q_0 + \frac{p_0}{m}t + \frac{at^2}{2}$. $\quad p = p_0 + at$

Thus A': $q' = q_A + \frac{p_A}{m}t + \frac{a}{2}t^2 \quad p'_A = p_A + at$; B': $q'_B = q_B + \frac{p_A}{m}t + \frac{at^2}{2}$. $p'_B = p_A + at$

C': $q'_C = q_B + \frac{p_C}{m}t + \frac{at^2}{2} \quad p'_C = p_C + at$

D': $q'_D = q_A + \frac{p_C}{m}t + \frac{at^2}{2} \quad p'_D = p_C + at$



Area I = Area II.

Thus Area of ABCD = A'B'C'D'.