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A general physics laboratory investigation of the thermodynamics of a rubber band

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The appropriateness of the standard equation of state of a rubber band is studied in an undergraduate laboratory experiment. There are two parts to the experiment: In the first part, a band is stretched at constant temperature; in the second, the band is kept at constant length while heated. The two coefficients in the equation of state of the rubber band are determined in four different ways: numerical differentiation of the data, numerical integration of the data, regression analysis, and linear regression on a "straight-line" form of the equation of state. The advantages and disadvantages of each technique are discussed. The characteristic constants describing the rubber band appear to vary by up to 15% depending on the technique of analysis employed. The origins of these variations are discussed.

I. INTRODUCTION

We have developed an undergraduate laboratory experiment in which the thermodynamic properties of an ordinary rubber band are measured. The experiment is simple and easily performed by one student, taking approximately 2 h to collect the data. This experiment shows that a rubber band follows a predicted equation of state over a restricted region of applied force and temperature which is chosen *a posteriori* to minimize possible deformation of the band and hysteresis. This range depends on the particular rubber band studied and hence the student must either perform the experiment twice (once to find this region, and once to obtain data) or the laboratory instructor must determine this range and state it as an experimental constraint.

This experiment has three goals: (1) to give the student the opportunity to measure the equation of state of a real substance using simple techniques; (2) to teach some of the techniques of data analysis, especially those related to "straight-line" analysis; (3) to help the student learn to work within the confines of simple apparatus and techniques. There are several aspects of this experiment, such as our measurement technique for the length of the rubber band, and even our choice of using a rubber band rather than a sheet of rubber, that could easily be changed. The philosophy of this laboratory is similar to that discussed in Baird's text.[1] There are several places where we try to implement ideas from his book. Most important, our focus is not strictly on obtaining the "correct" value for the coefficients in the equation of state, but rather on showing several ways to determine them and demonstrating the relative advantages and disadvantages of each technique.

An equation of state of a rubber band is[2]

$$F = AT(L/L_0 - L_0^2/L^2), \tag{1}$$

where A is a constant to be determined, L_0 is the length of the rubber band at zero applied force, F is the force on the rubber band causing it to stretch to lengths greater than L_0, T is the absolute temperature of the rubber band, and L is the length of the band when a force F is applied.

II. MEASUREMENT OF A AT CONSTANT TEMPERATURE

In the first part of this experiment, we determined the constant A at constant temperature by measuring the length of a thin rubber band as a function of applied force. The band used in the present experiment has an unstretched length $L_0 = 5.52$ cm and thickness 1 mm. In practice, L_0 is difficult to measure accurately. In the present experiment we estimated $L_0 = 5.50$ cm by direct measurement and found $L_0 = 5.52$ cm from the analysis given below. Figure 1 shows the apparatus used for this part of the experiment. The band was hung at the top from a thin, stable support rod of radius 2.5 mm, and a hook of radius 2 mm (mass = 4.1 g) was hung at the bottom. This hook allowed additional mass to be suspended from the band. L is measured as shown in the figure.[3] Experimentation showed that the best results are obtained when masses of less than 150 g are hung on the rubber band. Larger masses resulted in permanent deformation of the band and hysteresis.

Mass was added to the band in increments of 5 g. The length of the band was measured using a vernier caliper. The force on the rubber band was calculated as $F = mg$, where g is the magnitude of the acceleration of gravity. Room temperature was measured to be 21.2 °C (294.4 K). A graph of typical data obtained in this part of the experiment is shown in Fig. 2. The best straight line through the data and the least-squares fit of the data to Eq. (1) described below are also shown. Note that the data are not well represented by a straight line.

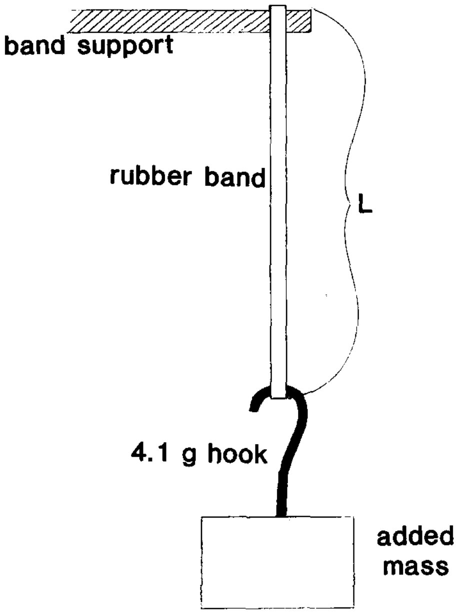

Fig. 1. Diagram of the apparatus used to determine F vs L.

The constant A can be determined from these data in several different ways. The most straightforward, although certainly not the simplest, technique is to perform a least-squares fit of the experimental data to Eq. (1). However, the second term on the right side of Eq. (1) is proportional to $1/L^2$; thus the normal fitting programs on calculators, etc. cannot be used for this purpose. We have written[4] a least-squares-fitting program that does fit the data directly to Eq. (1) and obtain from this program $A = 486 \pm 10$ dyn/K, and $L_0 = 5.52 \pm 0.04$ cm.

A second technique for analyzing the data consists of calculating the work done on the rubber band in stretching it from the length L_1 to L_2. From Eq. (1) it follows that

$$W = \int_{L_1}^{L_2} F\,dL = AT\left[\frac{L_2^2 - L_1^2}{2L_0} + L_0^2\left(\frac{1}{L_2} - \frac{1}{L_1}\right)\right]. \qquad (2)$$

The only unknown parameter in this equation is A. Hence, by numerically calculating the work (by determining the area under the F vs L curve) and comparing it to the right-hand side of Eq. (2), one can obtain A. Using this technique and all our data from $L_1 = 5.62$ cm to $L_2 = 8.31$ cm, with $L_0 = 5.52$ cm, we find $A = 485 \pm 25$ dyn/K.[5] However, knowledge of L_0 is critical to this calculation. The use of $L_0 = 5.56$ cm in this calculation yields $A = 500$ dyn/K; similarly, using $L_0 = 5.48$ cm yields $A = 470$ dyn/K. Even though the F vs L curve is not linear, a better estimate of L_0 than taking $L_0 = L$ when $m = 4.1$ g (the mass of the hook) can be obtained by using linear extrapolation back to $F = 0$ from the first two data points, for which the first term on the right side of Eq. (1) dominates the second term. In the present experiment this technique yields $L_0 = 5.56$ cm.

A third technique for analyzing the data consists of calculating

$$\left.\frac{\Delta F}{\Delta L}\right|_T = \frac{F_i - F_j}{L_i - L_j} \simeq \left.\frac{\partial F}{\partial L}\right|_T, \qquad (3)$$

where i and j refer to two neighboring data points shown in Fig. 2. Here, since the force is a state variable and the initial and final temperatures of the band are the same, to good approximation the measurement is performed isothermally. More complicated expressions may be used to calculate this derivative.[6] The experimentally determined value for $(\partial F/\partial L)_T$ is then compared to the value calculated for the equation of state, namely,

$$\left.\frac{\partial F}{\partial L}\right|_T = \frac{AT}{L_0} + \frac{2ATL_0^2}{L^3}. \qquad (4)$$

This form follows the slope–intercept form of a straight line, $y = mx + b$, with $x = 1/\langle L\rangle^3$, where $\langle L\rangle = (L_i + L_j)/2$ is the mean length of the rubber band between the two data points used to calculate $(\Delta F/\Delta L)_T$. Since $2ATL_0^2$ is a constant at constant temperature, the graph of $\Delta F/\Delta L$ vs $\langle L\rangle^{-3}$ should be a straight line. A graph of the experimentally determined $\Delta F/\Delta L$ vs $\langle L\rangle^{-3}$ is shown in Fig. 3. A standard least-squares fit to the data indicate that the slope is $(1.73 \pm 1.0)\times 10^7$ dyn cm^2 and the intercept is $(0.16 \pm 3.9)\times 10^4$ dyn/cm. From these data A is calculated in two ways: from the slope, $A = \text{slope}/2TL_0^2 = 920 \pm 530$ dyn/K; from the intercept,

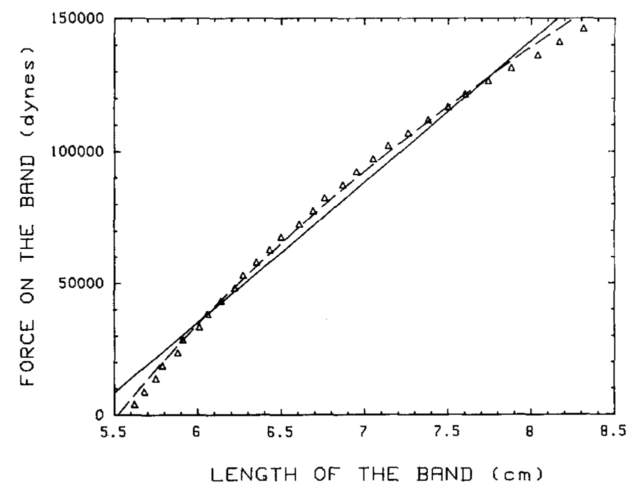

Fig. 2. Graph of typical data obtained using the apparatus shown in Fig. 1: (– – –) least-squares fit to Eq. (1) and (———) least-squares fit to a straight line, i.e., Hooke's law.

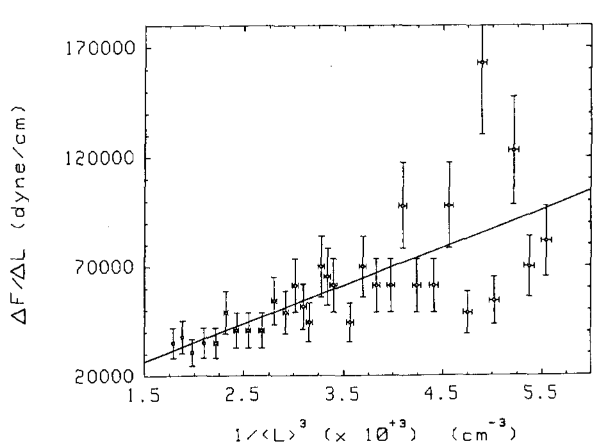

Fig. 3. Graph of experimentally determined $\Delta F/\Delta L$ vs $1/\langle L\rangle^3$. Typical error bars and the best straight line through the data are shown. The noise inherent in numerical differentiation is apparent.

$A = (\text{intercept})L_0/T = 31 \pm 730$ dyn/K. The uncertainty in the determination of A using this technique is quite large, which indicates that one should not use this technique. The reason is that numerical differentiation of the data introduces a great deal of noise into the determination of the coefficient A.

Also note that the data in Fig. 3 appear to deviate most from the least-square straight line at large $1/\langle L^3 \rangle$. This corresponds to small $\langle L \rangle$, where small uncertainties in the length lead to greater uncertainties in the values of $(\Delta F/\Delta L)$. However, an analysis of the data shown in Fig. 3 without the points with $1/\langle L \rangle^3$ greater than 4.25×10^{-3} yields, from the slope, $A = 872 \pm 300$ dyn/K, and from the intercept, $A = 27 \pm 700$ dyn/K, which is not an improvement. It is the noise introduced by numerical differentiation that is the origin of the large error in these results.

Another straight-line form of analysis is suggested by the equation

$$(L/L_0)^2 F = AT[(L/L_0)^3 - 1], \qquad (5)$$

which follows directly from Eq. (1). A graph of $(L/L_0)^2 F$ vs $(L/L_0)^3 - 1$ is shown in Fig. 4. The data shown in this figure were obtained using the value $L_0 = 5.50$ cm, as determined by measurements of the straightened out but unstretched band. Upon performing linear least-squares analysis on these data, we find that the slope is $(1.33 \pm 0.06) \times 10^5$ dyn, which corresponds to $A = 473 \pm 20$ dyn/K. The intercept should be 0; least-squares analysis shows the intercept to be 800 ± 2000 dyn. Clearly this is another useful technique for determining A. The primary weakness of this technique is that it does not directly yield a value for L_0. Thus L_0 must be determined independently. An analysis of the data using $L_0 = 5.52$ cm yields $A = 476 \pm 20$ dyn/K.

The use of all four types of analysis is instructive. The ease of using a least-squares-fitting program to find both of the parameters A and L_0 is balanced against the relative difficulty of writing the program. The smoothing effects of integration and the noise inherent in numerical differentiation are clearly demonstrated. While these latter two effects are well known in the literature,[7,8] this experiment demonstrates these effects very convincingly. In particular, the problems inherent in numerical differentiation of the data are evident from the large uncertainties in A obtained using this technique. Finally, the utility of another straight-line form of analysis that does not determine both A and L_0, but only A, is demonstrated. This last technique yields a value of A in good agreement with the values obtained by direct least-squares fitting of the data, and by calculating the work done in stretching the rubber band.

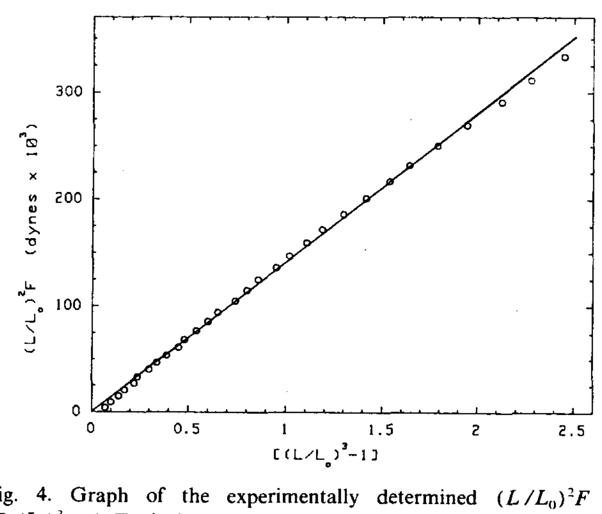

Fig. 4. Graph of the experimentally determined $(L/L_0)^2 F$ vs $(L/L_0)^3 - 1$. Typical error bars and the best straight line through the data are shown.

III. MEASUREMENT OF A AT CONSTANT LENGTH

In the second part of this experiment, we study $(\partial F/\partial T)_L$. From the equation of state,

$$\frac{1}{L}\frac{\partial F}{\partial T}\bigg|_L = \frac{A}{L_0} - \frac{AL_0^2}{L^3}. \qquad (6)$$

Since AL_0^2 is a constant, the graph of $1/L(\Delta F/\Delta L)_L$ vs $1/L^3$ is expected to be a straight line.

We obtain the data for this portion of the experiment using the setup shown in Fig. 5. A mass of 200 g is placed on a triple beam balance and attached to the rubber band as shown. By setting the balance off equilibrium, the stretching force exerted by the mass is shared by the band and the balance, thereby letting the force the mass exerts on the band be adjusted. The procedure followed is to pick a force, zero the balance, and then measure the length of the rubber band and the initial temperature, then heat the band and measure the final temperature and force.

The temperature was measured using a thermocouple, which has the advantage of being small, rugged, and easy to position close to the band. Thermocouples are discussed in many introductory thermodynamic books. Further details about their use is available from several sources.[9]

After the initial length and temperature of the band are measured, the band is heated to about 50 °C. At higher temperatures the rubber band may permanently deform; this might also vary for different types of bands and should be verified for the particular type of band studied. The bal-

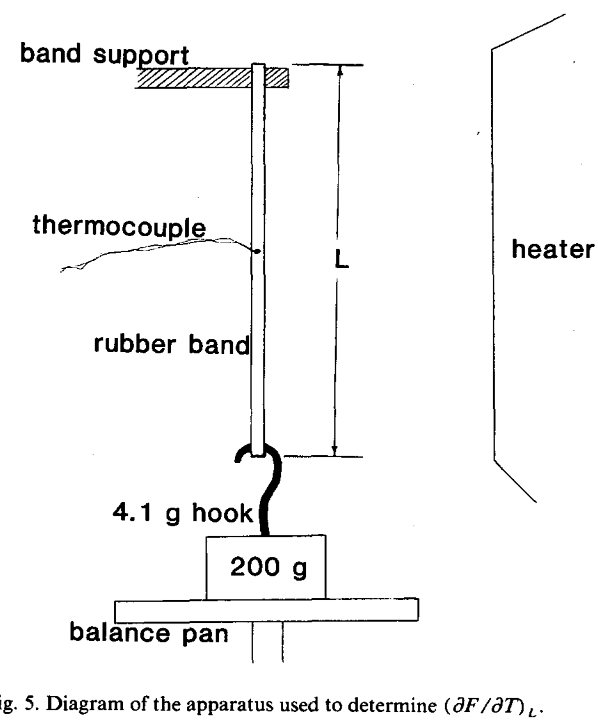

Fig. 5. Diagram of the apparatus used to determine $(\partial F/\partial T)_L$.

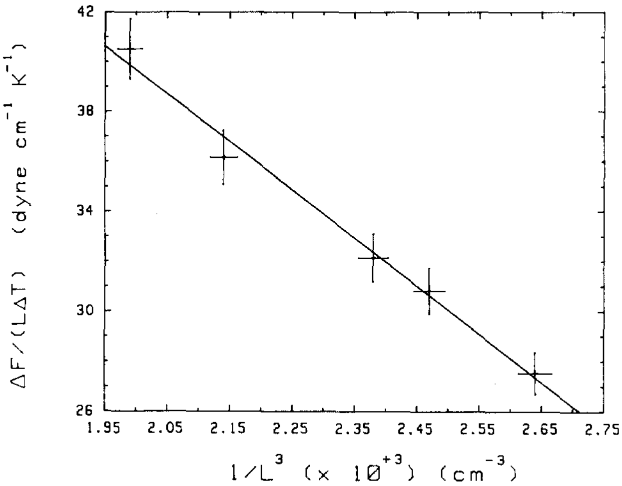

Fig. 6. Graph of experimentally determined $1/L(\Delta F/\Delta T)$ vs $1/L^3$.

ance is again zeroed to determine the change in force (through $F = mg$) caused by the band's "desire" to contract. The heater used must be one that heats the band uniformly. A heating lamp will not work if it is placed too close to the band. The authors recommend a typical household space heater to heat the rubber band. Once again a vernier caliper is used to measure the length of the band. Since the mass is placed at the center of the balance pan and the pan is zeroed at the start and end of the experiment, the change in length of the band is negligible. This procedure is repeated several times for initial forces in the same range as in the first part of the experiment. Figure 6 shows a graph of $1/L(\Delta F/\Delta T)$ vs $1/L^3$. The data indeed fit a straight line; a least-squares fit to the data indicates that the slope is $(-19.2 \pm 4) \times 10^3$ dyn cm^2 K^{-1} and the intercept is 78 ± 10 dyn cm^{-1}K^{-1}. With these data we calculate A from both the slope and the intercept and find $A = -(\text{slope})/L_0^2 = 610 \pm 120$ dyn/K and $A = (\text{intercept}) \times L_0 = 435 \pm 60$ dyn/K.

These values of A differ from the value obtained using the more reliable of the techniques described in Sec.II by from 9% to 15%. Nevertheless, when error bars are included, the values of A obtained using techniques 1, 2, and 4 from Sec. II and the present technique all agree. The rather large errors obtained using the present technique are largely the result of taking differences in two rather large quantities, the force and the temperature, and then dividing one difference by the other. The error in T is largely minimized by making the temperature change as large as possible. However, the error in F is largely determined by the uncertainty of the balance, and to an undetermined amount by nonuniform heating of the band, and possible small changes in its length.

IV. DISCUSSION

One of the purposes of the experiment we have considered is to teach various techniques for analyzing experimental data to obtain the parameters of interest. Therefore, ultimately one must ask "Which value of A is most trustworthy, and why?"

We would judge that the best value is obtained from the least-squares fit directly to Eq. (1), i.e., method 1. This value is obtained without performing any operations, such as multiplication, differentiation, or integration of the raw data, before fitting the data to a theoretical form. Also, this technique yields an independent measurement of the length with zero applied force. Further, the errors involved in obtaining A and in its value are well defined and known.

Our second choice is the integration technique, i.e., method 2. As discussed in the literature,[8] this technique tends to average out the errors in the raw data. A weakness of this technique of analysis is that the resultant value of A is strongly dependent on L_0, as is evident from Eq. (2). Thus an independent and accurate determination of L_0 must be made. Since most rubber bands do not have straight, roughly parallel sides when there is no applied force, it is difficult to obtain an accurate value for L_0.

A close tie for the second best technique is fitting the modified data to a straight-line form, i.e., method 4. This technique has two potential weaknesses. First, it does not determine L_0. Then, since L_0 occurs in the straight-line form to which the data are fitted, the value found for A depends on the value chosen for L_0. Second, because the force is multiplied by $(L/L_0)^2$, there is an unavoidable increase in the uncertainty of the fitted dependent variable. In fact, for forces determined to an accuracy only a few times better than those in the present experiment, the error in the independent variable will be comparable to or larger than the error in the dependent variable and a least-squares fit with errors in both coordinates will be necessary.[10]

Our fourth choice is measuring the temperature dependence of the force at constant length. The purpose of this section of the experiment was to verify the temperature dependence of the force when the length of the band is constant. As we discussed in Sec. III, there are inherently larger errors in determining A using this technique than in all the techniques considered in Sec. II save differentiation of the data.

As we discussed earlier, numerical differentiation of the raw data is not a good method of analysis for this experiment. Nevertheless, the various values of A are, to within experimental error, self-consistent. Further, the rubber band used is a good approximation of an ideal elastic substance, and, to good approximation, obeys the theoretical equation of state, Eq. (1). Finally, we note that there is negligible hysteresis—the effects of temperature and applied force on the rubber band are reproducible and repeatable, provided the band is not overheated or overstretched.

VI. CONCLUSION

This experiment presents the student with an opportunity to verify the equation of state of a common rubber band, study techniques for obtaining experimental parameters from data, and explore the thermodynamics of an everyday object.

ACKNOWLEDGMENTS

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[1] D. C. Baird, *Experimentation: An Introduction to Measurement Theory and Experiment Design* (Prentice-Hall, Englewood Cliffs, NJ, 1988),

2nd ed., pp. 1–6.

[2] See, for instance, M. W. Zemansky and R. H. Dittman, *Heat and Thermodynamics* (McGraw-Hill, New York, 1981), pp. 46 and 454; L. E. Reichl, *A Modern Course in Statistical Physics* (University of Texas Press, Austin, 1980), p. 56; T. L. Hill, *An Introduction to Statistical Thermodynamics* (Dover, New York, 1986), Chaps. 13 and 21.

[3] There are two important points to note. First, to avoid unduly large local stresses in the band, the support and hook should have a diameter of at least about five times the thickness of the band. Initial attempts to perform this experiment using a paper clip as the hook did not work. Second, this technique yields an approximate value of L. Upon the assumption that the band slides without sticking on both the support and hook, a more exact expression is

$$L = L_{SF} + (R_H + R_S)(\pi/2 - 1) + t(\pi/2 - 2),$$

where L is the midthickness point of the band to midthickness point of the band distance (average of inner and outer perimeters), L_{SF} is our expression for the length (the distance between the supports plus twice the thickness of the band), R_H and R_S are the radii of the hook and support, respectively, and t is the thickness of the band. To study the effects of a systematic error in L on A, we ran the data obtained in this experiment with both L_{SF} and this expression for L on a problem-solving program (EUREKA: THE SOLVER; Borlands, Scotts Valley, CA). Defining the difference between these two approximate measures of length as ΔL (i.e., $\Delta L \equiv L - L_{SF}$), we found $A = 500, 507, 514$ dyn/K for $\Delta = 0$, 0.1, and 0.2 cm, respectively. Thus an error in defining the length of the band affects both L_0 and A. Given the philosophy of this experiment, the magnitude of the systematic error introduced into the determination of A by the use of L_{SF} is acceptable.

[4] This is simply performed as a multiple linear regression with L and $1/L^2$ as variables. The authors will supply a copy of their BASIC program to interested parties.

[5] The major source of error came from the uncertainty in the two lengths used to calculate an element of work. It is critical that these lengths be determined as accurately as possible.

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Novel method for WKB analysis of multidimensional systems[a)]

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The availability of powerful microcomputers with graphics capabilities expands the range of numerical procedures that can be employed for undergraduate use. The use of simple software packages for the evaluation of multidimensional WKB approximate eigenvalues is discussed. Introductory courses in quantum mechanics rarely make use of semiclassical theories, such as WKB, to relate quantum mechanics to classical mechanics. This relationship can be helpful for the student's understanding of quantum mechanics, and a computer experiment can provide a useful avenue for making the concepts concrete. The major components of the approach involve the use of a package for the integration of differential equations, to treat the classical equations of motion, and a computer-aided design package to determine the phase integral areas. The codes are not expensive, and all exploit the graphical capabilities of microcomputers, which are critical for visualizing the relationship between classical and quantum ideas.

I. INTRODUCTION

For many years, the WKB approximation,[1] also called the semiclassical method, has been one of the standard methods for determining the eigenvalues of the Schrödinger equation when the potential energy function does not have a simple form. Almost all standard textbooks on quantum mechanics discuss the method as applicable to systems with large mass or large quantum numbers. During the past 10 years, a number of methods have been developed to calculate WKB eigenvalues for multidimensional nonseparable Hamiltonian systems.[2] One of the most useful and practical of the exact multidimensional WKB methods is the surface of section (SOS) method,[3] which computes the needed integrals from Poincaré surfaces of section. One purpose of this article is to introduce this

