# PS-523

Astrophysics, Gravitation and Cosmology

Part 3: Astrophysics

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# Lecture 1 : The interior solution: Oppenheimer-Volkoff equation

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The Schwarzschild solution expresses the gravitational field of a finite symmetric mass distribution *outside* it. There is a result, known as the *Birkhoff theorem*, which says that the field of a static spherically symmetric mass distribution in vacuum will always be exactly the Schwarzschild type, regardless of what kind of distribution is there inside as long as it is spherically symmetric.

We now look at the interior solution where $T^{\mu\nu}$, assumed to be as for a perfect fluid

$$T^{\mu\nu} = pg^{\mu\nu} + (\rho + p/c^2)U^\mu U^\nu$$

is not zero.

If it is static, then the metric is of the form

$$-a(r)(dx^0)^2 + b(r)(dr)^2 + r^2(d\theta)^2 + r^2\sin^2\theta(d\phi^2),$$

and the velocities $U^\mu$ of fluid trajectories are just

$$U^\mu = (c/\sqrt{a}, 0, 0, 0).$$

The stress-energy tensor corresponding to it is

$$\begin{aligned} T^{\mu\nu} &= \text{diagonal}\left(\frac{\rho c^2}{a}, \frac{p}{b}, \frac{p}{r^2}, \frac{p}{r^2\sin^2\theta}\right) \\ T_{\mu\nu} &= \text{diagonal}\left(\rho c^2 a, pb, pr^2, pr^2\sin^2\theta\right) \end{aligned}$$

where $\rho$ is the mass density and $p$ the pressure.

The various tensors for the Schwarzschild have been calculated before:

$\Gamma^0_{01} = \Gamma^0_{10} = (\ln a)'/2$

$\Gamma^1_{00} = a'/2b, \qquad \Gamma^1_{11} = (\ln b)'/2, \qquad \Gamma^1_{22} = -r/b, \quad \Gamma^1_{33} = -r\sin^2\theta/b$

$\Gamma^2_{12} = \Gamma^2_{21} = 1/r, \qquad \Gamma^2_{33} = -\sin\theta\cos\theta$

$\Gamma^3_{23} = \Gamma^3_{32} = \cot\theta, \qquad \Gamma^3_{13} = \Gamma^3_{31} = 1/r$

$$\begin{aligned} R_{00} &= \frac{a}{2b}\left[A'' + \frac{1}{2}A'(A' - B') + \frac{2A'}{r}\right] \\ R_{11} &= -\frac{1}{2}\left[A'' + \frac{1}{2}A'(A' - B') - \frac{2B'}{r}\right] \\ R_{22} &= -\frac{1}{b}\left[\frac{r}{2}(A' - B') + 1 - b\right] \\ R_{33} &= -\frac{\sin^2\theta}{b}\left[\frac{r}{2}(A' - B') + 1 - b\right] \end{aligned}$$

where we use the short hand $a = \exp A$ and $b = \exp B$.

The remaining $R_{\mu\nu}$ are zero. The curvature scalar is

$$R = -\frac{A''}{b} - \frac{A' - B'}{b}\left[\frac{A'}{2} + \frac{2}{r}\right] + \frac{2}{r^2}\left(1 - \frac{1}{b}\right)$$

and finally, the Einstein equation

$$\begin{aligned}
G_{00} &= \frac{ab'}{b^2 r} + \frac{a}{r^2}\left(1 - \frac{1}{b}\right) = \frac{8\pi G}{c^4}\rho c^2 a \\
G_{11} &= \frac{a'}{ar} + \frac{1}{r^2}(1 - b) = \frac{8\pi G}{c^4} bp \\
G_{22} &= \frac{r^2}{2b}\left[2\left(\frac{a'}{a}\right)' + \left(\frac{a'}{a} + \frac{2}{r}\right)\left(\frac{a'}{a} - \frac{b'}{b}\right)\right] = \frac{8\pi G}{c^4} r^2 p
\end{aligned}$$

The equation for $G_{33}$ is the same as for $G_{22}$.

The $G_{00}$ Einstein equation can be written after canceling out $a$, and multiplying by $r^2$

$$\left[r\left(1 - \frac{1}{b}\right)\right]' = \frac{8\pi G}{c^2}\rho r^2$$

which integrates to

$$r\left(1 - \frac{1}{b}\right) = \frac{2G}{c^2}\int_0^r 4\pi r^2 \rho(r) dr.$$

This determines $b$ to be of the same form as the vacuum Schwarzschild metric component

$$b = \left(1 - \frac{2Gm(r)}{rc^2}\right)^{-1}.$$

where

$$m(r) = \int_0^r 4\pi r^2 \rho(r) dr.$$

The $G_{11}$ equation then determines $A' = a'/a$ as

$$A' = 2G\,\frac{m + 4\pi r^3 p/c^2}{r(rc^2 - 2mG)}.$$

The third equation involves double primes, or second derivatives and will give us the variation of pressure or density with $r$.

In practice it is much easier to use the covariant divergence of $T^{\mu\nu}$ equal to zero, which after all is a consequence of the Bianchi identity. So we use

$$J^\mu \equiv T^{\mu\nu}{}_{;\nu} = T^{\mu 0}{}_{;0} + T^{\mu 1}{}_{;1} + T^{\mu 2}{}_{;2} + T^{\mu 3}{}_{;3} = 0$$

in place of the $G_{33}$ equation.

$$\begin{aligned}
T^{\mu 0}{}_{;0} &= \Gamma^0_{01} T^{\mu 1} + \Gamma^\mu_{00} T^{00} \\
T^{\mu 1}{}_{;1} &= T^{\mu 1}{}_{,1} + \Gamma^1_{11} T^{\mu 1} + \Gamma^\mu_{11} T^{11} \\
T^{\mu 2}{}_{;2} &= \Gamma^2_{21} T^{\mu 1} + \Gamma^\mu_{22} T^{22} \\
T^{\mu 3}{}_{;3} &= \Gamma^3_{31} T^{\mu 1} + \Gamma^3_{32} T^{\mu 2} + \Gamma^\mu_{33} T^{33}.
\end{aligned}$$

Of the four equations $J^\mu = 0$ three are trivial $0 = 0$, and only one non-trivial.

$$\begin{aligned} J^0 &= 0, \\ J^1 &= \frac{1}{2b}\left(2p' + A'(p + \rho c^2)\right) = 0 \\ J^2 &= 0 \\ J^3 &= 0 \end{aligned}$$

The nontrivial equation determines the rate of change of pressure as

$$p' = \frac{dp}{dr} = -G\frac{(\rho c^2 + p)(m + 4\pi r^3 p/c^2)}{r(rc^2 - 2mG)}$$

which can be written more transparently as

$$\frac{dp}{dr} = -\frac{Gm(r)}{r^2}\rho\left[1 + \frac{p}{\rho c^2}\right]\left[1 + \frac{4\pi r^3 p}{m(r)c^2}\right]\left[1 - \frac{2m(r)G}{rc^2}\right]^{-1}$$

where the three square brackets show the relativistic corrections to the Newtonian equation for equilibrium for a small area $\Delta A$

$$\Delta A dp = -(\rho \Delta A dr)\frac{Gm(r)}{r^2}.$$

This equation can be numerically integrated starting from $r = 0$. Choose the central pressure $p(0)$ and therefore the density $\rho(0)$ from the equation of state, and progressively calculate $p$ and $\rho$ which are related by the equation of state. At the boundary $r = R$of the star $p = 0$, and $\rho$ remains zero then on and the Schwarzschild solution takes over. See that the correction factors due to relativity tend to increase the magnitude of pressure gradient, making the star size (determined by $p = 0$) smaller. For a given equation of state, increasing central pressure may make the size of the star as small as 9/8 of the Schwarzschild radius, which is what happens for a neutron star. Beyond that a star may become a black hole.

# Lecture 2 : Fermi pressure

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When a star runs out of its 'nuclear fuel',

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it can become an equilibrium star, or, it may collapse endlessly

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If supported by electron fermi pressure
it is a 'white dwarf'

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If supported by neutron fermi pressure
it is a neutron star.

The usual star like our Sun starts its life when a vast amount of gas, mainly Hydrogen, falls under gravity and becomes heated up. Hydrogen burns into helium by the "p-p chain":

$$\begin{aligned} p + p &\rightarrow d + e^+ + \nu \\ p + d &\rightarrow {}^3He + \gamma \\ 2\ {}^3He &\rightarrow {}^4He + 2p \end{aligned}$$

As long as there are conditions favoring nuclear reactions for the formation of heavier nuclei with release of energy, the star may keep radiating and forming a core of heavier nuclei. But the binding energy curve shows that beyond iron, this is theoretically not possible to produce energy this way.

A star becomes a 'white dwarf' when the the equilibrium against gravity is not thermal pressure but the electron degeneracy pressure of its core of heavier elements. Such stars are small in size (a few thousand kilometers), mass of the order of a solar mass, have a very high surface temperature, that is why they appear 'white'.

It is not necessary for a star to reach the iron end point. Depending on the initial mass and other parameters it may follow other paths.

Roughly speaking, stars upto a few solar masses become white dwarfs, whereas heavier than 10 solar masses may become neutron stars through stages of red-giants or supernova explosions.

A white dwarf will have a core of high density with electrons free and not bound to nuclei. As electrons are fermions, they exert pressure even at temperature equal to absolute zero! The reason for this is that not more than one electron can be in a quantum state and even at absolute zero there will be states filled unto the Fermi level.

If there are $N$ electrons in a cubical box of side $L$, then its its discrete momenta take values

$$\mathbf{p} = \hbar \mathbf{k} = \hbar \frac{2\pi}{L}(n_1, n_2, n_3)$$

where $(n_1, n_2, n_3)$ are integers. There are thus two (for spin) quantum states In the $k$ space per $(2\pi/L)^3$ volume. The $N$ particles fill up$k$-space from zero up to $|\mathbf{k}| = k_F$ where

$$N = 2 \times \frac{4\pi}{3} k_F^3 \times \left(\frac{L}{2\pi}\right)^3 = \frac{k_F^3}{3\pi^2} L^3.$$

The momentum $p_F = \hbar k_F$ is called the Fermi momentum. The above expression can be written in terms of the number density $n = N/L^3$

$$n = \frac{k_F^3}{3\pi^2}.$$

At absolute zero the states are filled up from the ground state upwards. Even at absolute zero the Fermi momentum may be high enough if the density is high enough.

To calculate pressure, we can reason as follows. If we reduce the size of the box slightly, the states become more spread out in $k$-space (or momentum space). The same number $N$ will now of up to a higher value of the Fermi momentum. The total energy (sum of all energies upto the Fermi level) will also increase. This will be the work done against pressure.

$$p = -\left.\frac{\partial E}{\partial V}\right|_N.$$

It is preferable to use the number density $n = N/V$ in place of the volume variable $V$ and energy density $\rho = E/V$. As

$$\frac{\partial}{\partial V} = -\frac{N}{V^2}\frac{\partial}{\partial n},$$

we can write (with the understanding that $N$ is kept constant )

$$-p = \frac{\partial E}{\partial V} = \frac{\partial(\rho V)}{\partial V} = \rho + V\frac{\partial \rho}{\partial V} = \rho - n\frac{\partial \rho}{\partial n}.$$

If we know the dependence of the energy density $\rho$ on the number density $n$, we can find the *equation of state* giving pressure as a function of energy density.

For the extreme cases of non-relativistic (NR) and ultra relativistic (R) particles the energy momentum relations allow us to calculate energy density.

For NR case: $mc^2 + \hbar^2 k^2/2m$. There are $2L^3 d^3k/(2\pi)^3$ states in $d^3k$ volume in k-space. The energy in the NR case for all the occupied states is therefore

$$E = 2\frac{L^3}{8\pi^3}\int_0^{k_F} 4\pi k^2 [mc^2 + \hbar^2 k^2/2m] dk$$

which gives using the relation of $k_F$ to $n$

$$\rho = nmc^2 + \frac{3\hbar^2}{10m}(3\pi^2)^{2/3} n^{5/3} \qquad (NR).$$

Similarly for ultra high relativistic case where energy is simply $\hbar kc$ (the rest mass is neglected compared to kinetic energy),

$$\rho = \frac{3}{4}(3\pi^2)^{1/3}\hbar c n^{4/3} \qquad (R).$$

From these the pressure can be calculated.

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\begin{aligned}
p &= \frac{1}{5}(3\pi^2)^{2/3}\frac{\hbar^2}{m}n^{5/3} \qquad (NR) \\
p &= \frac{1}{4}(3\pi^2)^{1/3}\hbar c n^{4/3} \qquad (R)
\end{aligned}
$$

Once we have the equation of state, it is possible to integrate numerically the Oppenheimer-Volkoff equations starting from the central density and pressure. We continue integration and check if the solution corresponds to a finite value of $r = R$ where pressure goes to zero. That value also determines the value of the mass of the star. Thus for each value of $\rho(0)$ there is a value of $R$ as well as $M$. A plot of these is shown for different values of $\rho(0)$. For about $10^{11}$ g/cm$^3$ the mass is around 1.4 solar masses and the configuration becomes unstable: the electron pressure is unable to hold. This is the *Chandrasekar limit* for white dwarfs.