SU(2) as a Simplest Lie Algebra

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Abstract

In this lecture the group SU(2) is discussed with an eye on illustrating the abstract Lie algebra theory.

§1 Introdcution

In this lecture I will present some details of the Lie group $SU(2)$ which are necessary for understanding of the applications of general Lie algebra theory that has been presented.

This aim of this lecture is to

- identify what is that we want to learn in a course like this in Lie algebra? For this purpose I will remind you of what we know for SU(2).
- to explain the structure of a large class of Lie algebras CSLA (compact simple real Lie algebras). A Lie algebra in this class can be viewed as consisting of several Lie algebras of SU(2).
- to illustrate the applications for which the general theory of Lie algebra may be used.
- use SU(2) as an example to explain different concepts and techniques in the general theory of Lie algebra to be presented in later lectures.

§2 The Group

The first thing that we should ask is which group are we talking about? A group can be specified in several ways. For example the group U(3) can be specified as a set of all linear transformations on three complex numbers z_1, z_2, z_3

$$(z_1, z_2, z_3) \Longrightarrow (z_1', z_2', z_3') \tag{1}$$

which leaves $|z_1|^2 + |z_2|^2 + |z_3|^2$ invariant, *i.e.*

$$|z_1|^2 + |z_2|^2 + |z_3|^2 = |z_1'|^2 + |z_2'|^2 + |z_3'|^2. \tag{2}$$

In the initial stages it will be useful to work with matrix groups

This is the first thing that you may want to know when you hear about SU(2) or SU(3)? Obviously you will ask

Question: What is SU(2) and what is SU(3)? SU(2) is a group of 2×2 unitary matrices having determinant 1 and SU(3) is a group of 3×3 unitary matrices having determinant 1.

Question for you Show that the set of of linear transformations satisfying (5.2) coincides with the set of 3×3 unitary matrices.

Note: *Remember to prove both the following statements*

(i) a transformation by a unitary matrix obeys (5.2), and

(ii) start from (5.2) and prove that the linear transformation must be given by a unitary matrix.

The first part is easy, but the second one is non trivial.

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§3 Lie algebra of a Lie group

Next we come to Lie algebra. *You may ask, "What is a Lie algebra?"*

Definition 1 *A real Lie algebra $\mathscr{L}$ is a real vector space and $\forall u, v \in \mathbb{L}$ a bracket $[u, v]$ is defined. This bracket is again an element of Lie algebra and has all the properties of commutator of two matrices.*

The vectors space dimension of a Lie algebra will be called its *dimension* , denoted by r, equals the number of parameters required to specify the group elements.

You may want to see an example. So let us ask, "What is the Lie algebra of group SU(2)?"

Given a group like SU(2), we arrive at its Lie algebra by looking at the structure of group elements in a neighbourhood of identity. How do we do this for SU(2)? When I say SU(2), I do not mean SU(2) only, you should be able to find the Lie algebra of any of the matrix groups; in fact we should keep an eye on concepts and techniques that can be carried over to an arbitrary Lie group.

So let us have a 2×2 unitary matrix with determinant 1 as U. Then

$$U^\dagger U = UU^\dagger = I, \qquad \det U = 1. \tag{3}$$

For Lie algebra we need to look at the elements of the group close to the identity. So let U be infinitesimally close to the identity. So write it as $U = I + \delta U$ and impose the above two requirements?

$$U^\dagger U = I \;\; \Rightarrow \;\; (I + \Delta U^\dagger + \Delta U) \approx I \Rightarrow \Delta U^\dagger = -\Delta U \tag{4}$$

$$\det U \;\; \approx \;\; (1 + \mathrm{trace}(U)) \Rightarrow \mathrm{trace}(U) = 0. \tag{5}$$

(Give a proof of the above statement.)

Thus we see that the Lie algebra of SU(2) consists of all 2×2 anti-hermitian, complex, traceless matrices.

§4 Structure constants

Now that we have Lie algebra, we can talk about the *structure constants* of the Lie group. *So you may ask," What are structure constants?"* You may also want to know about some important properties of structure constants.

A Lie algebra is a vector space and so we can select basis $\{e_1, e_2, e_3\}$ in the vector space and the Lie brackets of the basis elements

$$[e_j, e_k] = c_{jk}^m e_m \tag{6}$$

define the structure constants c_{jk}^m.

In any vector space basis is not unique, infinitely many choices are possible. Obviously the structure constants depend on the choice of basis. *So it is natural to ask whether the structure constants can "simplified" ? or by a suitable choice of basis, can the structure constants be brought to some standard from?* This is where the theorems of Lie Cartan and Weyl help us in guaranteeing that the answer is yes for a large class of Lie algebras (CSLA) and that such a convenient choice of basis, used extensively, does exist.

In particular, it is known that for SU(n), O(N), Sp(2n) etc. we can choose the basis so as to make the structure constants completely antisymmetric. The next question for you gives basis in which the structure constants of SU(2) become antisymmetric.

The three 2×2 matrices X_k a basis in the Lie algebra of SU(2) and satisfy the commutation relations

$$[X_j, X_k] = \epsilon_{jkm} X_m \tag{7}$$

and ϵ_{jkm} is the Levi Civita symbol.

If we repeat the above analysis for the rotation group $SO(3)$ (or of $O(3)$) in three dimensions, we get the the set of 3×3 real antisymmetric matrices as the Lie algebra of $SO(3)$. Here again we can find basis vectors, three 3×3 matrices, which give rise to the same structure constants as in the above equation and we say that the Lie algebra of the rotation group is identical to that of SU(2).

A elements of a basis in a Lie algebra are called *generators* of the Lie algebra. It must be emphasised that the matrices X_k are merely an example of commonly used choice of generators.

Questions For You:

[1] Obtain the Lie algebra of O(N) and find its dimension. $[Ansr = N(N-1)/2]$

[2] Verify that every element of SU(2) Lie algebra has three independent real parameters and can be written as

$$u = \sum_{k=1}^{3} a_k X_k, \qquad \text{and where } X_k = i\sigma_k, k = 1, 2, 3 \tag{8}$$

where $a_k, k = 1, 2, 3$ are real and $\sigma_k, k = 1, 2, 3$ are the three Pauli matrices.

[3] Is the above basis unique? Or is there another basis for which the structure constants are the same?

In Physics literature we work with unitary representations and hermitian generators. Thus the generators X_k get replaced by $T_k \equiv iX_k$ for the case of isospin, by spin operators $S_k \equiv iX_k$ in case of applications to spin and by angular momentum operators $J_k \equiv iX_k$ in case of rotation group in 3 dimensions.

The above procedure of arriving at the Lie algebra can be used to arrive at the Lie algebra of any given group for example O(3), SU(3), Sp($2n$) etc. If the group is defined as a set of transformation on some space, it will be useful to reformulate that group as a matrix group. For example for this purpose, it will be convenient to regard the group of rotations in three dimensions as the matrix group SO(3). Then the Lie algebra of the rotation group in three dimensions can be obtained by the above procedure.

§5 Cartan Subalgebra

From the generators we come to the Cartan sub algebra. The Cartan sub algebra is the maximal set of commuting generators H_a. In case of SU(2), out of the three generators we can take any one generator in this set. For SU(N) this set will consist of $(N-1)$ elements. The number of such elements is called the *rank of the Lie algebra*.

The generators in Cartan sub algebra have a direct physical interpretation. These $H's$ correspond to quantum numbers of physical states. For example in case of SU(3) there are two elements in the Cartan sub algebra and these for flavour SU(3) are identified with the z- component of isospin, I_z and hypercharge Y.

In case of SU(2) the rank is 1 and there is only one generator in the Cartan sub algebra which can be taken as the 3rd component of isospin.

§6 Root vectors

In SU(2) the diagoal generator is usually taken as T_3, the third component of isospin. The other two generators can be combined to form the raising and lowering operators $T_\pm = T_1 \pm iT_2$. We then have the following commutation relations

$$[T_3, T_\pm] = \pm T_\pm. \tag{9}$$

In case of a CSLA Lie algebra, there are $(r-\ell)$ raising and "lowering generators", denoted by $E_{\pm\alpha}, \alpha = 1, 2, \ldots, (r-\ell)/2$. The corresponding commutation relations are

$$[H_a, E_{\pm\alpha}] = \pm\alpha_a E_{\pm\alpha} \tag{10}$$

The components of a root vector are in one to one correspondence with H_a. The set of numbers $\{\alpha_a, a = 1, 2, \ldots, \ell\}$ are components of "root vector". For a fixed α we collect α_a appearing in Eq.(10) for all different H_a, that is a root vector $\overline{\alpha}$.

In SU(2) the rank is 1 and there is only one H_a, so the root vector is one component object. In all there are two root vectors corresponding to the generators $T_\pm$.

The values of components of permissible root vectors determine the "change in values" of H_a under action of $E_{\pm\alpha}$. For most case, see theorem in later lecture, the $E_{\pm\alpha}$ satisfy

$$\Big[E_\alpha, E_{-\alpha}\Big] = \alpha_a H_a \tag{11}$$

which for SU(2) takes the form

$$\Big[T_+, T_-\Big] = 2T_z. \tag{12}$$

The root space is one dimensional for SU(2). There are two root vectors ± 1. These determine the change in values of 3rd component T_3 under action of the ladder operators $T_\pm$.

Questions for You:

[1] What is the dimension of root space for SU(2)? What are the components of different root vectors?

[2] In case of SU(3) the "hermitian" generators satisfy relations that can be cast in the form [?]

$$[T_3, Y] = 0; \tag{13}$$
$$[T_3, T_\pm] = \pm \qquad [Y, T_\pm] = 0 \tag{14}$$
$$[T_3, U_\pm] = \mp\frac{1}{2}U_\pm, \qquad [Y, U_\pm] = \pm U_\pm \tag{15}$$
$$[T_3, V_\pm] = \pm\frac{1}{2}V_\pm, \qquad [Y, V_\pm] = \pm V_\pm \tag{16}$$
$$[T_+, T_-] = 2T_3, \quad [U_+, U_-] = \frac{3}{2}Y - T_3, \qquad [V_+, V_-] = \frac{3}{2}Y + T_3, \quad \text{etc} \tag{17}$$

Find the root vectors for SU(3) this basis. Represent them as vector in a plane choosing T_3 and Y as axes. Show the action of $T_\pm, U_\pm, V_\pm$ on a root vector in generating a new root vector.

Representations

What is that we want to know for applications to Physics? For very many applications we need representations of the Lie algebra.

So the question is what is representation? A representation comes with a vector space and operators in the vector space. If the vector space is finite dimensional the operators become matrices when a basis is used. The vectors in the representation space give possible states of physical systems. There must be operator which are in a one to one correspondence with elements of the Lie algebra. Eigenvectors of Cartan subalgebra are usually chosen as basis. The Lie algebra commutation relations determine the action of generators on the basis and hence this information is sufficient to construct the matrices for the generators.

It is sufficient to work with irreducible representations (IRR), and all other representations can be constructed from IRR.

States and Operators in an IRR of Spin Algebra,

Let us restate the above requirements in terms of spin. We can think of the SU(2) algebra as algebra of spin operators. In this case H is like S_z and the ladder operators are $S_\pm$. For a particle with spin, we want to know

- *What are allowed values of total spin $\vec{S}^2$?*
  Recall that $\vec{S}^2$ commutes with all the three components of spin operators and can have values $s(s+1)\hbar^2$.
- *Given a particular value of s, what are the allowed values (eigenvalues of) S_3?*
  The answer is that for a given spin s, the 3rd component can have values $(2s+1)$ given by $m = s, s-1, \ldots, -s$.

A particle with spin s, the wave function will have $(2s+1)$ components, each component giving the probability amplitude for the 3rd component having a particular value. For example for a spin 3/2 particle the spin part of wave function will look like

$$\chi = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} \tag{18}$$

and different $|c_k|^2$, for $k = 1, 2, 3, 4$, respectively give the probabilities of S_3 being $3/2, 1/2, 1/2, -3/2$ in units of $\hbar$.

Next we want to know how the spin operators S_1, S_2, S_3 are going to act on a wave function like this? They will become some matrices and will have dimension $(2s+1)$; they will have $(2s+1)$ rows and $(2s+1)$ columns. So you need to know how to construct these matrices? These matrices will give a $(2s+1)$ dimensional representation of the SU(2) algebra. The matrices can be constructed by

- first computing the matrix elements of $S_\pm = S_1 \pm iS_2$ which in Dirac notation are given by
  $$\langle m_1 | S_\pm | m_2 \rangle$$
  and can be calculated using the raising and lowering action of the operators $S_\pm$:
  $$S_\pm |m\rangle = \sqrt{s(s+1) - m(m\pm 1)}\,\hbar\,|m\pm 1\rangle. \tag{19}$$

Specifying an IRR

For SU(2) the dimension of the matrices can have any integral value.

But in general this will not be the case. Therefore, we would also like to know what are allowed values of the dimensions of a given Lie algebra? For this purpose it is sufficient to work out everything for "irreducible representations"(IRR). How do I specify an IRR?

- One way to specify an IRR is to give values of *all Casimir operators.* A Casimir operator is an operator constructed out of the elements of Lie algebra, which commutes with all the generators. For example for SU(2) it is the operator S^2 and its allowed values are $s(s+1)\hbar^2$, with s taking all positive integral and half integral values.

- Another way to specify an IRR representation of SU(2) is to specify the highest value that S_3 can take, for spin s the highest value of S_3 is $m = s$.

  For an IRR of a general Lie algebra, the set $\{H_a\}$ will play the role of S_3, the eigenvalues of H_a will be called "weight vectors" and highest weights (or equivalently a set of integers) can be used to specify an IRR.

Some special representations of SU(2)

In case of SU(2) we already know two representations. We have seen that the Lie algebra of SU(2) consists of 2×2 anti-hermitian matrices. A basis set of 2×2 matrices is $\frac{i}{2}\sigma_k, k = 1, 2, 3.$

$$ie_k \longrightarrow \frac{1}{2}\sigma_k$$

(1/2) is included to normalize them in a standard fashion. The corresponding representation space is $\mathbb{C}^2$, consisting of two component complex column vectors.

Another representation of SU(2) is the adjoint representation already mentioned. This is present for every Lie algebra and is defined in terms of the structure constants. In the adjoint representation we have

$$e_j \longrightarrow X_j, \text{ with } (X_j)_{km} = c^k_{jm}. \tag{20}$$

where the structure constants for SU(2) can be taken to be $c^k_{jm} = \epsilon_{jkm}$.

Questions for You:

1. What are the dimensions of the two representations of SU(2) given above?

2. What spin values the two representations of SU(2) of spin algebra correspond to?

3. What will be the dimension of the adjoint representation of SU(3)? for any Lie algebra.

If we start from a state $|jm\rangle$ of spin j and S_z value m, applying S_+ several times, say p times, will lead to the highest value (equal to j) of S_3 and further application of the raising operator will give 0. Similarly, application of the lowering operator S_- several times, say q, times will lead to a state with lowest value $-j$ of S_3 and further application of S_- will give a null vector. Thus we see that

$$m + p = j, \text{ and } m - q = -j \Longrightarrow 2m + p - q = 0. \tag{21}$$

Since p, q are integers, we get the result that $m = n/2$, where n is an integer.

We remark that all representations of SU(2) can be constructed out of the spin 1/2 representation by a process of taking tensor product of representation 1/2.

Tensor Operators:

We need the concept of irreducible tensor operators of SU(2). A tensor operator of rank k is a set of $(2k+1)$ operators $T \equiv \{T_q^k | q = k, k-1, \ldots, -k\}$ such that they obey commutation relations

$$[J_\pm, T_q^{(k)}] \;=\; \sqrt{k(k+1) - q(q\pm 1)}\hbar T_{q\pm 1}^{(k)}. \tag{22}$$

So the action of commutator with $J_\pm$ is to shift value of q by one unit. Again there is a highest value of q after n_1 steps and and a lowest value of q after n_2 steps, Then by an argument similar to that given above, q itself must an $n/2$ for some integer n. The if we select $\mathrm{SU}(2)^{(\alpha)}$ for a particular α, all the other generators are tensor operators w.r.t. the $\mathrm{SU}(2)^{(\alpha)}$ for each α.

General Lie Algebras

Let us now see how the above discussion get carried over for a general Lie algebra. For the purpose of generalization of SU(2) to any CSLA we should keep the following correspondence in mind.

$$S_3 \to \{H_1, H_2, \ldots\}, \qquad S_\pm \to \{E_{\pm\alpha}, E_{\pm\beta}, \ldots\}. \tag{23}$$

The methods available for SU(2) become become applicable to a general CSLA because for each root α we can define a SU(2) subalgebra by introducing

$$J_\pm = \frac{\sqrt{2}}{|\alpha|} E_{\pm\alpha}, \qquad J_3 = \frac{\alpha \cdot H}{|\alpha|^2}. \tag{24}$$

These objects satisfy the Lie algebra of SU(2), to be denoted as $\mathrm{SU}(2)^{(\alpha)}$. If β is any other root, $\beta \neq \alpha$, one can easily verify the following commutation relations.

$$[J_3, E_\beta] = m E_\beta, \qquad \text{with } m = \frac{\alpha \cdot \beta}{|\alpha|^2} \tag{25}$$

and the set of operators $\left\{E_{\beta+p\alpha}, E_{\beta+(p-1)\alpha}, \cdots, E_{\beta-q\alpha}\right\}$ behave like tensor operators of some rank j. Using the argument given above, we get a useful result that

$$m = \frac{\alpha \cdot \beta}{|\alpha|^2} \tag{26}$$

should be equal to $(n_1 - n_2)/2$, *i.e.*

$$\frac{\alpha \cdot \beta}{|\alpha|^2} = \frac{n}{2}, \tag{27}$$

where n is a positive or negative integer. The above process can be repeated with roles of α and β interchanged giving

$$\frac{\alpha \cdot \beta}{|\beta|^2} = \frac{n'}{2}, \tag{28}$$

where, again, n' is a positive or negative integer.

These relations put restrictions on the allowed roots of a Lie algebra and are important for classification of CSLA. We will not have time to go through details of this important work on classification of CSLA.