Notes For Lectures in Quantum Mechanics *

Induced Emission and Absorption

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1 Weak field approximation

In Coulomb gauge the Hamiltonian for an electron in an atom is described by $H' = H_1 + H_2$, where

$$H_1 = \frac{e}{mc}\vec{A}\cdot\vec{p}, \qquad H_2 = \frac{e^2}{2mc}\vec{A}^2. \tag{1}$$

It will be seen that the term H_2 is of higher order as compared to H_1 and can be ignored in the first order calculations.

2 The Poynting Vector

The plane wave solution for the vector potential $\vec{A}(\vec{r},t)$ can be written as

$$\vec{A}(\vec{r},t) = \vec{A}_0 \exp\big(i\vec{k}\cdot\vec{r} - i\omega t\big) + \text{c.c.} \tag{2}$$

*qm-lec-25004; Updated:Jul 17, 2021; Ver 0.x where c.c. means complex conjugate. Taking $\vec{A}_0 = |\vec{A}_0|e^{i\alpha}$, the expression vector potential in ((2)) can also be written as

$$\vec{A}(\vec{r},t) = |\vec{A}_0(\omega)| \cos\left(i\vec{k}\cdot\vec{r} - i\omega t\right) \tag{3}$$

The Poynting vector $\vec{N} = \frac{c}{4\pi}\left(\vec{E}\times\vec{B}\right)$ relates the amplitude $|A_0(\omega)|$ the energy density per unit frequency. The energy in the frequency range $\omega, \omega + \Delta\omega$ is in fact given by

$$I(\omega) = \frac{\omega^2}{2\pi c}|\vec{A}_0^{\,2}|\Delta\omega \tag{4}$$

3 The matrix element

If we write $\vec{A}_0 = |\vec{A}_0|\hat{e}$,where $\hat{e}$ gives the direction of polarization, the matrix element appearing in Eq.(**??**) is then written as

$$\langle f|H_1|i\rangle = |\vec{A}_0|\langle f|\exp(i\vec{k}\cdot\vec{r})\hat{e}\cdot\vec{p}|i\rangle \tag{5}$$

4 Dipole approximation

For atomic transitions in the lowest order it is a good approximation to replace $\exp(i\vec{k}\cdot\vec{r})$ by 1. Thus the matrix element becomes

$$\langle f|H_1|i\rangle = |\vec{A}_0|\langle f|\hat{e}\cdot\vec{p}|i\rangle \tag{6}$$

Next we note the commutator of $\hat{e}\cdot\vec{p}$ is proportional to the commutator of $\hat{e}\cdot\vec{r}$ with the Hamiltonian $H_0 = \frac{\vec{p}^2}{2m} + V(\vec{r})$

$$(\hat{e}\cdot\vec{r})H_0 - H_0(\hat{e}\cdot\vec{r}) = \frac{1}{m}\hat{e}\cdot\vec{p} \tag{7}$$

Taking matrix element between the two states $|i\rangle$ and $|f\rangle$ we get

$$\langle f|\hat{e}\cdot\vec{r}H_0|i\rangle - \langle f|\hat{e}\cdot\vec{r}H_0|i\rangle = i\hbar\frac{1}{m}\langle f|\hat{e}\cdot\vec{p}|f\rangle \tag{8}$$

Using the fact that $|i\rangle$ and $|f\rangle$ are eigenvectors of H_0 with eigenvalues E_i and E_f we get

Perturbation theory

The amplitude of transition from an initial state $|i\rangle$ to a final state $|f\rangle$ is given by

$$
\begin{aligned}
|C_{fi}(t)|^2 &= 4|\langle f|H'|i\rangle|^2 \frac{\sin^2 \frac{1}{2}(\omega_{fi}-\omega)t}{\hbar^2(\omega_{fi}-\omega)^2} \\
&\quad +4|\langle f|H'|i\rangle|^2 \frac{\sin^2 \frac{1}{2}(\omega_{fi}+\omega)t}{\hbar^2(\omega_{fi}+\omega)^2} \\
&\qquad +\text{Cross terms}
\end{aligned}
\tag{9}
$$

where $\omega_{fi} = (E_f - E_i)/\hbar$. The case $E_f > E_i$ with $\omega_f i - \omega \approx 0$ the first term is most important and this corresponds to absorption. The second case, when $E - f < E_i$ and $\omega_{fi} + \omega \approx 0$, correspond to emission of radiation. Both emission and absorption can take place in presence of radiation. The emission process in presence of external radiation is called *induced emission.*

Experimentally an atom in higher state can emit radiation and go to a state with lower energy even in absence of external radiation- a process known as **spontaneous emission**.

Given a pair of states $|m\rangle$ and $|n\rangle$, probabilities for the two transitions - induced emission and induced absorption - given by one of the two terms in Eq.(**??**) - are equal.

5 Induced emission transition probability

Following the same steps as in the derivation of Fermi Golden rule, the transition probability per unit time takes the form

$$
w_{fi} = \frac{4\pi e^2}{3\hbar^2 c} I(\omega) |\langle f|\hat{e}\cdot\vec{r}|i\rangle|^2 \tag{10}
$$

where an averaging over different direction has been done.

In those cases when the incident radiation is incoherent superposition of radiation of several frequencies, we add the probabilities for different frequencies, and the probability of transition after time t will be given by

$$
P_{fi}(t) = \int_0^t |C_{fi}(t)|^2 \, d\omega \tag{11}
$$

6 A limitation of semiclassical theory

The semi-classical theory however does not explain why an atom in higher state, emits radiation and goes to a state with lower energy even in absence of external radiation – a process known as spontaneous emission. This probability can be computed only when the radiation field is quantized. However, Einstein showed how an answer can be obtained using Planck's law of black body radiation.