Notes for Lectures in Quantum Mechanics *

Electormagnetic Waves

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Contents

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1 Maxwell's Equations

We shall first discuss the interaction of a charged particle with electromagnetic waves in the classical Maxwell's theory. Classically the electromagnetic fields are described by the four Maxwell's equations which written in cgs units take the form

$$\nabla \cdot \vec{E} = 4\pi\rho \tag{1}$$

$$\nabla \times \vec{E} = -\frac{d\vec{B}}{dt} \tag{2}$$

$$\nabla \cdot \vec{B} = 0 \tag{3}$$

$$\nabla \times \vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{d\vec{E}}{dt} \tag{4}$$

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Here ρ is the charged density and $\vec{J}$ is the current density which satisfy the continuity equation

$$\frac{d\rho}{dt} + \nabla \cdot \vec{J} = 0 \tag{5}$$

2 Free E.M. waves

The Maxwell's equations admit freely propagating wave solutions even in absence of external charges and currents. Setting $\rho = 0, \vec{J} = 0$ in Eq.(1)-((4)), we get

$$\begin{aligned} \nabla \cdot \vec{E} &= 0 && (6) \\ \nabla \times \vec{E} &= -\frac{d\vec{B}}{dt} && (7) \\ \nabla \cdot \vec{B} &= 0 && (8) \\ \nabla \times \vec{B} &= \frac{1}{c}\frac{d\vec{E}}{dt} && (9) \end{aligned}$$

and the wave equations for the electric and magnetic fields follwo from these equations. The equations are

$$\begin{aligned} \nabla^2 \vec{E} &= \frac{1}{c^2}\frac{d^2\vec{E}}{dt^2} && (10) \\ \nabla^2 \vec{B} &= \frac{1}{c^2}\frac{d^2\vec{B}}{dt^2} && (11) \end{aligned}$$

3 Scalar and vector potential

The Eq.(3) suggests that there exists a vector potential $\vec{A}$ such that the magnetic field can be written as

$$\vec{B} = \nabla \times \vec{A} \tag{12}$$

Eq.(2) and (12) then imply that

$$\nabla \times \left(\vec{E} + \frac{d\vec{A}}{dt}\right) = 0 \tag{13}$$

This shows that there exists a scalar function ϕ such that

$$\vec{E} + \frac{d}{dt}\vec{A} = -\nabla\phi \tag{14}$$

Thus the electric field can be written as

$$\vec{E} = -\nabla\phi - \frac{d}{dt}\vec{A} \tag{15}$$

4 Gauge transformation and gauge invariance

It is easy to see that the potentials ϕ and $\vec{A}$ are not uniquely determined by the fields $\vec{E}$ and $\vec{B}$. In fact two sets of potentials $\phi, \vec{A}$ and $\phi', \vec{A}'$ related by

$$\phi' = \phi - \frac{d\vec{\Lambda}}{dt} \tag{16}$$

$$\vec{A}' = \vec{A} + \nabla\Lambda \tag{17}$$

give the same answers for the electric and the magnetic fields where Λ is an arbitrary function of space time. Since all observable quantities are functions of the electric and magnetic fields, they are also the same for the two sets of potentials. Thus for example the Lorentz force

$$\vec{F} = q\big(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\big) \tag{18}$$

experienced by a charged particle in electric and magnetic field does not depend on the choice of the function Λ in Eq.(16) and ((17)). The transformation $(\vec{A}, \phi) \to (\vec{A}', \phi')$, given by ((16)) is called a **gauge transformation** and we say that the observable quantities are invariant under gauge transformation.

5 Gauge condition

For a given set of electric and magnetic fields one has infinitely many possible answers for the potentials, corresponding to the different choices of the *gauge parameter* Λ. The choice of potentials can be restricted by requiring them to satisfy further condition(s), called **gauge condition**. A change in gauge condition amounts to making a gavuge transformation and is of no physical consequence.

When describing free e.m. waves, $\rho = 0, \vec{j} = 0$ it turns out to be convenient to choose the gauge condition

$$\phi = 0, \qquad \nabla \cdot \vec{A} = 0 \tag{19}$$

For such a choice of gauge condition, the two wave equations Eq.(10)-((11)) become equivalent to the wave equation

$$\nabla^2 \vec{A} = \frac{1}{c^2}\frac{d^2\vec{A}}{dt^2} \tag{20}$$

6 Plane wave solutions

Taking Eq.(19) as the gauge condition, the plave wave solutions of the wave equation ((17)) can be written as

$$\vec{A}(\vec{r},t) = \vec{A}_0 \exp(i\vec{k}\cdot\vec{r} - i\omega t) \tag{21}$$

where $\vec{k}$ is the propagation vector, $\omega = ck$ is the frequency and the amplitude vector $\vec{A}_0$ satisfies the transversality condition

$$\vec{k}\cdot\vec{r} = 0. \tag{22}$$