Let's Start Talking *

Spin Wave Function and Spin Operators

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1 Representation of Spin Wave Function

In order to describe the spin degrees of freedom, it is convenient to introduce a representation. For this we need to select a complete commuting set of hermitian operators and construct an orthonormal basis from their simultaneous eigenvectors. A suitable set consists of $\vec{S}^2$ and S_z. In order to proceed further, we want to work with an explicit representation of the spin. We arrange the eigenvectors $|s, m\rangle$ in *descending order* in m to get a basis $\big\{|s, m\rangle \big| m = s, s-1, \cdots, -s+1, -s\big\}$. An arbitrary state vector $|x\rangle$ is then a linear combination of the basis elements

$$|x\rangle = \sum_{m=-s}^{s} \alpha_m |sm\rangle \tag{1}$$

The interpretation of the numbers α_m is that square of its modulus, $|\alpha_k|^2$, gives the probability that S_z will have the corresponding value $m\hbar$.

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Following the convention of arranging the basis vectors in the order of decreasing values for the spin projection S_z, the $|x\rangle$ will be represented by a column vector

$$\chi = \begin{pmatrix} \alpha_s \\ \alpha_{s-1} \\ \vdots \\ \alpha_{-s} \end{pmatrix} \tag{2}$$

with $(2s+1)$ components.

2 Representation of Spin Operators

The spin operators $\vec{S}$ will be represented by matrices with $(2s+1)$ rows and $(2s+1)$ columns. First of all, the matrix for S_z will be diagonal matrix with eigenvalues of S_z appearing along the main diagonal.

$$S_z = \hbar \begin{pmatrix} s & 0 & 0 & \cdots & \cdots & 0 \\ 0 & s-1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & s-2 & \cdots & \cdots & 0 \\ 0 & 0 & 0 & \cdots & \cdots & \\ \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & 0 & \cdots & \cdots & -s \end{pmatrix} \tag{3}$$

The matrices for S_x and S_y are found by first obtaining the matrices for $S_\pm$ and using $S_x = \frac{1}{2}(S_+ + S_-)$ and $\frac{-i}{2}(S_+ - S_-)$. To construct these matrices one needs to know the matrix elements $\langle s, m'|S_\pm|s, m\rangle$ which can be computed by making use of the result

$$\boxed{S_\pm|s, m\rangle = \sqrt{s(s+1) - m(m \pm 1)}\,\hbar\,|s, m \pm 1\rangle} \tag{4}$$

3 Spin 1 matrices

The corresponding result for the spin one matrices is

$$S_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}; \quad S_y = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}; \quad S_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

and is left as en exercise for the reader.

4 Spin 1/2 matrices

We shall give the answer for spin $\frac{1}{2}$ and spin 1 matrices. The spin half matrices are related to the Pauli matrices $\sigma_x, \sigma_y, \sigma_z$ and are given by

$$\vec{S} = \frac{\hbar}{2}\,\vec{\sigma} \tag{5}$$

This result is derived in the solved problem below.