Notes for Lectures in Quantum Mechanics *

Validity of Born Approximation

For spherically symmetric potential $V(r)$

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The integral equation for the energy eigen functions was derived to be

$$\psi(\vec{r}) = e^{i\vec{k_i}\cdot\vec{r}} - \left(\frac{\mu}{2\pi\hbar^2}\right)\int \frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}V(\vec{r}')\psi(\vec{r}')d^3r\prime \tag{1}$$

In the derivation of the integral equation it is assumed that it is good approximation to take $\psi(\vec{r})$ to be plane wave

$$\psi(\vec{r}) \approx \exp(i\vec{k_i}\cdot\vec{r}) \tag{2}$$

and substitute in the right hand side of Eq.(1). The second term on the right hand side of Eq.(1)gives a correction to plane wave form. If Eq.(2) is a good approximation to the wave function, the correction must be small compared to the plane wave term. Hence the condition, under which the Born approximation is valid, is given by

$$\left|\left(\frac{\mu}{2\pi\hbar^2}\right)\int \frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}V(\vec{r}')\psi(\vec{r}')d^3r\prime\right| < \left|e^{i\vec{k_i}\cdot\vec{r}}\right| \tag{3}$$

or

$$\left|\left(\frac{\mu}{2\pi\hbar^2}\right)\int \frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}V(\vec{r}')\exp(i\vec{k_i}\cdot\vec{r}')d^3r\prime\right| < 1 \tag{4}$$

The effect of the potential is to distort the wave function and make it different from the plane wave and clearly this distortion is expected to be maximum where the potential is large. The potential is assumed to tend to zero as $r\to\infty$, assuming that the potential is of short range and that most significant effect comes for $r\approx 0$, we apply the condition Eq.(4)for $r=0$ and get

$$\left|\left(\frac{\mu}{2\pi\hbar^2}\right)\int \frac{e^{ikr'}}{r'}V(\vec{r}')\exp(i\vec{k_i}\cdot\vec{r}')d^3r\prime\right| < 1 \tag{5}$$

Changing the integration variable name from r' to r, assuming the potential to be spherically symmetric we get

$$\left(\frac{\mu}{2\pi\hbar^2}\right)\left|\int_0^{2\pi}d\phi\int_0^{\pi}\sin\theta d\theta\int_0^{\infty}\frac{e^{ikr}}{r}V(r)e^{ikr\cos\theta}r^2dr\right| < 1 \tag{6}$$

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The ϕ integration gives 2π and the θ integral is

$$\int_0^\pi \sin\theta d\theta e^{ikr\cos\theta} = \int_{-1}^1 \exp(ikrt)dt \tag{7}$$

$$= \frac{1}{ikr}\left(e^{ikr} - e^{-ikr}\right) \tag{8}$$

Substituting Eq.(8) in Eq.(6) we get

$$\frac{\mu}{\hbar^2 k}\left|\int_0^\infty \left(e^{2ikr} - 1\right) V(r)dr\right| < 1 \tag{9}$$