Notes for Lectures in Quantum Mechanics *

Born Approximation

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## 1 Perturbative solution of integral equation

The energy eigen functions, with a correct asymptotic behaviour corresponding to the scattering solutions satisfy the following integral equation

$$\psi(\vec{r}) = \exp(i\vec{k}_i.\vec{r}) - \frac{1}{4\pi}\int \frac{\exp\{ik|\vec{r}-\vec{r}'|\}}{|\vec{r}-\vec{r}'|}U(\vec{r}')\psi(\vec{r}')d^3r'. \tag{1}$$

where we have used the notation

$\vec{k}_i$ = momentum of the incident beam

$\vec{k}_f$ = momentum of the scattered beam

$\theta$= scattering angle = angle bewteen $\vec{k}_i$ and $\vec{k}_f$.

$V(\vec{r})$ = potential due the target

$\mu$= mass of the incident particle ( reduced mass for two body problem)

$U(\vec{r})= \dfrac{2\mu}{\hbar^2}V(\vec{r})$.

An iterative solution of the integral equation can be obtained by assuming that in the lowest order approximation $\psi(\vec{r})$ is equal to $\psi_0(\vec{r})$ given by

$$\psi_0(\vec{r}) = \exp(i\vec{k}_i\cdot\vec{r}) \tag{2}$$

Using this approximation for $\psi(\vec{r})$ from Eq.(2) in the right hand side of Eq.(1), we get the next order solution, denoted as $\psi_1(\vec{r})$, given by

$$\psi_1(\vec{r}) = \exp(i\vec{k}_i\cdot\vec{r}) - \frac{1}{4\pi}\int \frac{\exp\{ik|\vec{r}-\vec{r}'|\}}{|\vec{r}-\vec{r}'|}U(\vec{r}')\exp(i\vec{k}_i\cdot\vec{r}')d^3r'. \tag{3}$$

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*Updated:; Ver 0.x

The next approximation to the solution, $\psi_2(\vec{r})$, is obtained by replacing $\psi(\vec{r})$ in Eq.(1) with $\psi_1(\vec{r})$. Thus

$$\psi_2(\vec{r}) = e^{i\vec{k}_i.\vec{r}} - \frac{1}{4\pi}\int \frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}U(\vec{r}')\psi_1(\vec{r}')d^3r\prime \tag{4}$$

$$= e^{i\vec{k}_i\cdot\vec{r}} - \frac{1}{4\pi}\int \frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}U(\vec{r}')e^{(i\vec{k}_i\cdot\vec{r}')}d^3r\prime \tag{5}$$

$$+\frac{1}{(4\pi)^2}\int \frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}U(\vec{r'})\int \frac{e^{ik|\vec{r}'-\vec{r}''|}}{|\vec{r'}-\vec{r''}|}U(\vec{r}'')e^{i\vec{k}_i\cdot(\vec{r}'+\vec{r}'')}d^3r'd^3r''. \tag{6}$$

This process can be continued indefinitely and it becomes very cumbersome to compute the wave function beyond first few orders.

## 2 First Born Approximation

The first order Born approximation consists in using the first, the plane wave term in the above series as approximate wave function in the expression

$$f(\theta,\phi) = -\frac{1}{4\pi}\int \exp(-i\vec{k}_f\cdot\vec{r}')U(r')\psi(r')d^3r' \tag{7}$$

$$= -\left(\frac{\mu}{2\pi\hbar^2}\right)\int \exp(-i\vec{k}_f\cdot\vec{r}')V(r')\psi(r')d^3r' \tag{8}$$

for the scattering amplitude, giving

$$f(\theta,\phi) \approx -\frac{\mu}{2\pi\hbar^2}\int e^{i(\vec{k}_i-\vec{k}_f)\cdot\vec{r}}V(r)d^3r, \tag{9}$$

or

$$f(\theta,\phi) \approx -\frac{\mu}{2\pi\hbar^2}\int e^{-i\vec{q}\cdot\vec{r}}V(r)d^3r, \tag{10}$$

where $\vec{q} = \vec{k}_i - \vec{k}_f$ is the momentum transfer. The result Eq.(10) is the well known, first order, Born approximation result for the scattering amplitude.