Notes for Lectures in Quantum Mechanics *

Integral Equation for Scattering

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1 Green function for free particle Schrodinger Equation

In order to convert the Schrodinger equation

$$\left\{\frac{-\hbar^2}{2\mu}\nabla^2 + v(r)\right\}\psi = E\psi \tag{1}$$

into an integral equation we first rewrite it as

$$\left(\nabla^2 + k^2\right)\psi = u(\vec{r})\psi(\vec{r}) \tag{2}$$

where $k^2 = \dfrac{2\mu E}{\hbar^2}$, $U(r) = \dfrac{2\mu}{\hbar^2}V(r)$ and we have defined Green function $G(\vec{r})$ as a solution of

$$\left(\nabla^2 + k^2\right)G(\vec{r}) = -\delta(\vec{r}). \tag{3}$$

Two possible solutions for Green function are $G(\vec{r})$ are

$$G_\pm(\vec{r}) = \frac{e^{\pm ikr}}{4\pi r}. \tag{4}$$

Using the Green function we can now write down a "formal" solution of Eq.(2) as a solution of integral equation.

2 Set up integral equation

Using a Green function which is a solution of Eq.(3) a formal solution for Eq.(2) can be written as

$$\psi(\vec{r}) = \phi(\vec{r}) - \int G(|\vec{r}-\vec{r}\,'|)U(|\vec{r}\,'|)\psi(\vec{r}\,')d^3r' \tag{5}$$

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where $\phi(\vec{r})$ is a solution of the equation

$$\left(\nabla^2 + k^2\right)\phi(\vec{r}) = 0. \tag{6}$$

It turns out that the choice

$$G(\vec{r}) = \frac{e^{ikr}}{4\pi r} \tag{7}$$

for the Green function leads to the correct scattering boundary condition on the wave function.

3 Verify boundary condition

For the scattering problem we must select the function $\phi(\vec{r})$ to be

$$\phi(\vec{r}) = \exp(i\vec{k_i}.\vec{r}) \tag{8}$$

where $\vec{k_i}$ is the momentum of the incident particles. Substituting Eq.(7) Eq.(8) in Eq.(5) we get the integral equation for the scattering to be

$$\psi(\vec{r}) = \exp(i\vec{k_i}.\vec{r}) - \frac{1}{4\pi}\int \frac{e^{ik|\vec{r}-\vec{r'}|}}{|\vec{r}-\vec{r'}|}U(\vec{r'})\psi(\vec{r'})d^3r'. \tag{9}$$

To verify that $\psi(\vec{r})$ given by Eq.(9) does indeed have correct asymptotic property, we need to expand $|\vec{r}-\vec{r'}|$ in powers of $\frac{\vec{r}}{\vec{r'}}$ we shall assume that the potential is short range potential so that the contribution to integral over $\vec{r'}$ comes from small value of r'.

Some algebraic manipulations

Expand $|\vec{r}-\vec{r'}|$ in powers of r'

$$\begin{aligned} |\vec{r}-\vec{r'}| &= \sqrt{r^2 + r'^2 - 2\vec{r}\cdot\vec{r'}} && (10)\\ &= r\left(1 - 2\frac{\vec{r}\cdot\vec{r'}}{r^2} + \frac{r'^2}{r^2}\right)^{1/2}. && (11)\end{aligned}$$

Using binomial expansion we get

$$|\vec{r}-\vec{r'}| = r\left(1 - \frac{\vec{r}\cdot\vec{r'}}{r^2} + O\left(\frac{r'^2}{r^2}\right)^2\right). \tag{12}$$

Large r expansion of the formal solution (9)

We substitute Eq.(12) in the exponential and in the factor $\frac{1}{|\vec{r}-\vec{r'}|}$ in Eq.(9) and write $\frac{1}{|\vec{r}-\vec{r'}|} \approx \frac{1}{r}$ to get

$$\begin{aligned} \psi(\vec{r}) &\longrightarrow \exp(i\vec{k_i}\cdot\vec{r}) - \frac{1}{4\pi r}\int \exp\left(ikr - ik\frac{\vec{r}\cdot\vec{r'}}{r^2}\right)U(\vec{r'})\psi(\vec{r'})d^3r' && (13)\\ &= \exp(i\vec{k_i}\cdot\vec{r}) - \frac{e^{ikr}}{4\pi r}\int \exp(-ik\hat{n}\cdot\vec{r'})U(r')\psi(r')d^3r'. && (14)\end{aligned}$$

In the last step we have introduced a unit vector $\hat{n} = \vec{r}/r$. The Eq.(14) gives the probability amplitude (wave function) at $\vec{r}$. If the particles are to reach at a detector at $\vec{r}$, the vector $\hat{n}$ must be in the direction of the final momentum and parallel to k_f. Note that

$$|\vec{k}_i| = |\vec{k}_f| = k \tag{15}$$

holds as a consequence of energy conservation and hence

$$k(\hat{n}\cdot\vec{r}') = \vec{k}_f\cdot\vec{r}. \tag{16}$$

Thus Eq.(14) takes the form

$$\psi(\vec{r}) \approx \exp(i\vec{k}_i\cdot\vec{r}) - \frac{e^{ikr}}{4\pi r}\int \exp(-i\vec{k}\cdot\vec{r}')U(r')\psi(r')d^3r'. \tag{17}$$

This asymptotic behaviour is of the form expected for large r

$$\psi(\vec{r}) \approx \exp(i\vec{k}_i\cdot\vec{r}) - \frac{e^{ikr}}{r}f(\theta,\phi). \tag{18}$$

4 Scattering amplitude

Comparing Eq.(17) with Eq.(18) we see that the scattering amplitude is given by

$$\begin{aligned} f(\theta,\phi) &= -\frac{1}{4\pi}\int \exp(-i\vec{k}_f\cdot\vec{r}')U(r')\psi(r')d^3r' && (19)\\ &= -\left(\frac{\mu}{2\pi\hbar^2}\right)\int \exp(-i\vec{k}_f\cdot\vec{r}')V(r')\psi(r')d^3r'. && (20)\end{aligned}$$

It must be noted that the integral equation Eq.(9) and the expression for the scattering amplitude in Eq.(20) are **exact** results.