Lectures For Lectures in Quantum Mechanics*

Perturbative Solution of Differential Equation

Converting to an Integral Equation

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1 Setting up integral equation

Very often the problem of solving a linear differential equation can be replaced with solution of an integral equation. An important feature of the integral equation approach is that the initial conditions to be satisfied by the solution is built into the integral equation. The integral equation can be solved many a times by an iterative procedure which we shall illustrate by an example of a simple differential equation.

Suppose we are interested in solving the differential equation

$$\frac{dy}{dx} = \lambda y \tag{1}$$

subject to the boundary condition

$$y(x)|_{x=x_0} = N \tag{2}$$

where N is a constant. To convert Eq.(1) into an integral equation we integrate Eq.(1) to get

$$y(x) = \lambda \int y(x)dx + constant \tag{3}$$

or to be more precise, let us write Eq.(3) as

$$y(x) = \lambda \int_0^x y(t)dt + constant \tag{4}$$

The constant in the above equation is fixed by making use of the initial condition Eq.(2), and we get const= N and

$$y(x) = N + \lambda \int_0^x y(t)dt \tag{5}$$

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*Updated:; Ver 0.x

It is to noted that the unknown function y appears inside the integral sign, hence an equation of this type is called integral equation. This equation can be solved iteratively giving a solution as a series in powers of λ. This method gives the exact answer in the case of this simple example under consideration.

2 Perturbative Solution

As a first step, we set y equal to y_0 where

$$y_0 = N \tag{6}$$

is the solution in the zeroth order in λ and is simply taken to be equal to the first term Eq.(5). Next the zeroth order 'solution' ,y_0, is substituted in the right hand side of Eq.(5) to get the solution in the first order in λ. Thus we have

$$y_1(x) = N + \lambda \int_0^x y_0 dx \tag{7}$$

$$= N + \lambda \int_0^x N dx \tag{8}$$

$$\therefore \qquad y_1(x) = N(1+\lambda)x \tag{9}$$

To improve the approximation, we substitute $y_1(x)$ for $y(x)$ in right hand side of Eq.(2) to get the next approximation $y_2(x)$ for our solution.

$$y_2(x) = N + \lambda \int_0^x y_1(x) dx \tag{10}$$

$$= N + \lambda \int_0^x N(1+\lambda x) dx \tag{11}$$

$$= N\left(1 + \lambda x + \lambda^2 \frac{x^2}{2}\right) \tag{12}$$

continuing in this fashion we get

$$y_3(x) = N + \lambda \int_0^x y_2(x) dx \tag{13}$$

$$= N + N\lambda \int_0^x \left(1 + \lambda x + \lambda \frac{x^2}{2}\right) dx \tag{14}$$

$$= N\left(1 + \lambda x + \frac{\lambda^2 x^2}{2} + \frac{\lambda^3 x^3}{3!}\right) \tag{15}$$

$$y_4(x) = N\left(1 + \lambda x + \frac{\lambda^2 x^2}{2} + \frac{\lambda^3 x^3}{3!} + \frac{\lambda^4 x^4}{4!}\right) \tag{16}$$

Thus we get an infinite series in powers of λ

$$y(x) = N\left(1 + \lambda x + \frac{\lambda^2 x^2}{2} + \frac{\lambda^3 x^3}{3!} + \ldots\right) \tag{17}$$

summing the series we get

$$y(x) = Ne^{\lambda x} \tag{18}$$

Note that this is the correct solution for ordinary differential equation and satisfies the given boundary condition $y(0) = N$.

3 Alternate approach – Using series expansion

The above method of solution is equivalent to the following alternate sequence of steps. We want to solve

$$y(x) = N + \lambda \int_0^x y(t)dt \tag{19}$$

we assume that the solution can be written as a series in λ

$$y(x) = \alpha_0 + \lambda\alpha + \lambda^2\alpha^2 + \cdots + \alpha^n\lambda^n + \cdots \tag{20}$$

we substitute Eq.(20) in Eq.(19) and compare powers on both the sides. Therefore we get

$$\alpha_0(x) + \lambda\alpha_1(x) + \lambda^2\alpha_2(x) + \ldots \lambda^n\alpha_n(x) + \ldots \tag{21}$$

$$= N + \lambda \int_0^x \left\{\alpha_0(t) + \lambda\alpha_1(t) + \lambda^2(t)\alpha_2(t) + \cdots\right\} dt \tag{22}$$

on comparing coefficients of different powers of λ we successively get

$$\begin{aligned}
\alpha_0(x) &= N \\
\alpha_1(x) &= \int_0^x \alpha_0(t)dt = Nx \\
\alpha_2(x) &= \int_0^x \alpha_1(t)dt = N\frac{x^2}{2} \\
\alpha_1(x) &= \int_0^x \alpha_2(t)dt = N\frac{x^3}{3!} \\
\alpha_k(x) &= \int_0^x \alpha_{k-1}(t)dt = N\frac{x^k}{k!}
\end{aligned} \tag{23}$$

Therefore we get

$$y(x) = N\left(1 + \lambda x + \frac{\lambda^2}{2!}x^2 + \cdots + \frac{\lambda^n}{n!}x^n + \cdots\right) \tag{24}$$