Notes for Lectures on Quantum Mechanics *

Addition of Angular Momenta Using Tables

A. K. Kapoor
http://0space.org/users/kapoor
akkapoor@cmi.ac.in; akkhcu@gmail.com

We shall now take up an example of constructing the states $|JM\rangle$ using the tables of Clebsch Gordon coefficients. There are two tables of Clebsch Gordon coefficients given at the end. Note that the first table is for $j_2 = \frac{1}{2}$ and the second one for $j_2 = 1$. For $j_2 = \frac{1}{2}$, the two columns correspond to the two values, $\frac{1}{2}$ and $-\frac{1}{2}$, of m_2. The two rows correspond to the two possible values of total angular momentum $J = j_1 + \frac{1}{2}$ and $J = j_1 - \frac{1}{2}$. Similarly, the second table, corresponding to $j_2 = 1$, has three columns for the three values $m_2 = 1, 0, -1$ and the three rows correspond to three allowed values $J = j_1 + 1, j_1, j_1 - 1$ of total angular momentum.

Question: Construct all possible states with $J = \frac{5}{2}, \frac{3}{2}$ in terms of states with $j_1 = 2, j_2 = \frac{1}{2}$ using the table of Clebsch Gordon coefficients.

☺*Solution*: Since $j_2 = \frac{1}{2}$ the first table is needed here. The allowed values of J are $\frac{5}{2}$ and $\frac{3}{2}$. For all $J = \frac{5}{2}$ states we must use the first row of the first table and for $J = \frac{3}{2}$ states the second row should be used.

1. The state with highest values $J = \frac{5}{2}$ and $M = \frac{5}{2}$ can be written down directly as there is only one possible set of values $(m_1, m_2) = (2, 2; \frac{1}{2}, \frac{1}{2})$ is allowed. Hence

$$|\boldsymbol{\tfrac{5}{2}\tfrac{5}{2}}\rangle = |2, 2; \tfrac{1}{2}, \tfrac{1}{2}\rangle \tag{1}$$

2. For the next state with $J = \frac{5}{2}$ and $M = \frac{3}{2}$ the possible values of (m_1, m_2) are $(1, \frac{1}{2})$ and $(2, -\frac{1}{2})$ and hence

$$|\boldsymbol{\tfrac{5}{2}}, \boldsymbol{\tfrac{3}{2}}\rangle = \alpha_1|2, 1; \tfrac{1}{2}, \tfrac{1}{2}\rangle + \alpha_2|2, 2; \tfrac{1}{2} - \tfrac{1}{2}\rangle \tag{2}$$

and the coefficients α_1, α_2 are to be read from the first row.

$$\alpha_1 = \sqrt{\frac{j_1 + M + \frac{1}{2}}{2j_1 + 1}} = \sqrt{\frac{2 + \frac{3}{2} + \frac{1}{2}}{2.2 + 1}} = \sqrt{\frac{4}{5}}; \tag{3}$$

$$\alpha_2 = \sqrt{\frac{j_1 + M - \frac{1}{2}}{2j_1 + 1}} = \sqrt{\frac{2 - \frac{3}{3} + \frac{1}{2}}{2.2 + 1}} = \sqrt{\frac{1}{5}} \tag{4}$$

3. For the next state with $J = \frac{5}{2}$ and $M = \frac{1}{2}$ the possible values of (m_1, m_2) are $(0, \frac{1}{2})$ and $(1, -\frac{1}{2})$ and hence

$$|\boldsymbol{\tfrac{5}{2}}, \boldsymbol{\tfrac{3}{2}}\rangle = \alpha_3|20; \tfrac{1}{2}, \tfrac{1}{2}\rangle + \alpha_4|2, 1; \tfrac{1}{2}, -\tfrac{1}{2}\rangle \tag{5}$$

and the coefficients α_1, α_2 are again to be read from the first row.

$$\alpha_3 = \sqrt{\frac{j_1 + M + \frac{1}{2}}{2j_1 + 1}} = \sqrt{\frac{2 + \frac{1}{2} + \frac{1}{2}}{2.2 + 1}} = \sqrt{\frac{3}{5}}; \tag{6}$$

$$\alpha_4 = \sqrt{\frac{j_1 + M - \frac{1}{2}}{2j_1 + 1}} = \sqrt{\frac{2 - \frac{1}{2} + \frac{1}{2}}{2.2 + 1}} = \sqrt{\frac{2}{5}} \tag{7}$$

* qm-lec-17009 Updated:; Ver 0.x

4 For the next state with $J=\frac{5}{2}$ and $M=-\frac{1}{2}$ the possible values of (m_1, m_2) are $(-1, \frac{1}{2})$ and $(0, -\frac{1}{2})$ and hence

$$|\tfrac{\mathbf{5}}{\mathbf{2}}, -\tfrac{\mathbf{1}}{\mathbf{2}}\rangle = \alpha_5|2,-1;\tfrac{1}{2}\tfrac{1}{2}\rangle + \alpha_6|2,0;\tfrac{1}{2},-\tfrac{1}{2}\rangle \tag{8}$$

and the coefficients α_1, α_2 are again to be read from the first row.

$$\alpha_5 = \sqrt{\frac{j_1+M+\frac{1}{2}}{2j_1+1}} = \sqrt{\frac{2-\frac{1}{2}+\frac{1}{2}}{2.2+1}} = \sqrt{\frac{2}{5}}; \tag{9}$$

$$\alpha_6 = \sqrt{\frac{j_1+M-\frac{1}{2}}{2j_1+1}} = \sqrt{\frac{2+\frac{1}{2}+\frac{1}{2}}{2.2+1}} = \sqrt{\frac{3}{5}} \tag{10}$$

5 Next for state with $J=\frac{5}{2}$ and $M=-\frac{3}{2}$ the possible values of (m_1, m_2) are $(-2, \frac{1}{2})$ and $(-1, -\frac{1}{2})$ and hence

$$|\tfrac{\mathbf{5}}{\mathbf{2}}, -\tfrac{\mathbf{3}}{\mathbf{2}}\rangle = \alpha_7|2,-2;\tfrac{1}{2},\tfrac{1}{2}\rangle + \alpha_8|2,-1;\tfrac{1}{2},-\tfrac{1}{2}\rangle \tag{11}$$

and the coefficients α_1, α_2 are again to be read from the first row.

$$\alpha_7 = \sqrt{\frac{j_1+M+\frac{1}{2}}{2j_1+1}} = \sqrt{\frac{2-\frac{3}{2}+\frac{1}{2}}{2.2+1}} = \sqrt{\frac{1}{5}}; \tag{12}$$

$$\alpha_8 = \sqrt{\frac{j_1+M-\frac{1}{2}}{2j_1+1}} = \sqrt{\frac{2+\frac{3}{2}+\frac{1}{2}}{2.2+1}} = \sqrt{\frac{4}{5}} \tag{13}$$

6 Finally the state $|\frac{5}{2}, -\frac{5}{2}\rangle$ has the highest value of J and the lowest value of M, and easily written as

$$|\tfrac{\mathbf{5}}{\mathbf{2}}, -\tfrac{\mathbf{5}}{\mathbf{2}}\rangle = |2,-2;\tfrac{1}{2},-\tfrac{1}{2}\rangle \tag{14}$$

7 Now we take up the construction of states with $J=\frac{3}{2}$ and $M=\frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}$ we write

$$\begin{aligned}
|\tfrac{\mathbf{3}}{\mathbf{2}}, \tfrac{\mathbf{3}}{\mathbf{2}}\rangle &= \beta_1|2,1;\tfrac{1}{2},\tfrac{1}{2}\rangle + \beta_2|2,2;\tfrac{1}{2},-\tfrac{1}{2}\rangle \\
|\tfrac{\mathbf{3}}{\mathbf{2}}, \tfrac{\mathbf{1}}{\mathbf{2}}\rangle &= \beta_3|2,0;\tfrac{1}{2},\tfrac{1}{2}\rangle + \beta_4|2,1;\tfrac{1}{2},-\tfrac{1}{2}\rangle \\
|\tfrac{\mathbf{3}}{\mathbf{2}}, -\tfrac{\mathbf{1}}{\mathbf{2}}\rangle &= \beta_5|2,-1;\tfrac{1}{2},\tfrac{1}{2}\rangle + \beta_6|2,0;\tfrac{1}{2},-\tfrac{1}{2}\rangle \\
|\tfrac{\mathbf{3}}{\mathbf{2}}, -\tfrac{\mathbf{3}}{\mathbf{2}}\rangle &= \beta_7|2,-2;\tfrac{1}{2},\tfrac{1}{2}\rangle + \beta_8|2,-1;\tfrac{1}{2},-\tfrac{1}{2}\rangle
\end{aligned}$$

where $\beta_1, \beta_3, \beta_5, \beta_7$ are to be taken from the entries in the second row and first column. The coefficients $\beta_1, \beta_3, \beta_5, \beta_7$ are to read from the second row of the first table with appropriate values of M. Thus with $j_1 = 2$ we get

$$\beta_1 = -\sqrt{\frac{j_1-M+\frac{1}{2}}{2j_1+1}}\Bigg|_{M=\frac{3}{2}} = -\sqrt{\frac{1}{5}} \quad ; \quad \beta_2 = \sqrt{\frac{j_1+M+\frac{1}{2}}{2j_1+1}}\Bigg|_{M=\frac{3}{2}} = \sqrt{\frac{4}{5}}$$

$$\beta_3 = -\sqrt{\frac{j_1-M+\frac{1}{2}}{2j_1+1}}\Bigg|_{M=\frac{1}{2}} = -\sqrt{\frac{2}{5}} \quad ; \quad \beta_4 = \sqrt{\frac{j_1+M+\frac{1}{2}}{2j_1+1}}\Bigg|_{M=\frac{1}{2}} = \sqrt{\frac{3}{5}}$$

$$\beta_5 = -\sqrt{\frac{j_1-M+\frac{1}{2}}{2j_1+1}}\Bigg|_{M=-\frac{1}{2}} = -\sqrt{\frac{3}{5}} \quad ; \quad \beta_6 = \sqrt{\frac{j_1+M+\frac{1}{2}}{2j_1+1}}\Bigg|_{M=-\frac{1}{2}} = \sqrt{\frac{2}{5}}$$

$$\beta_7 = -\sqrt{\frac{j_1-M+\frac{1}{2}}{2j_1+1}}\Bigg|_{M=-\frac{3}{2}} = -\sqrt{\frac{4}{5}} \quad ; \quad \beta_8 = \sqrt{\frac{j_1+M+\frac{1}{2}}{2j_1+1}}\Bigg|_{M=-\frac{3}{2}} = \sqrt{\frac{1}{5}}$$

1 Tables of Clebsch Gordon Coefficients

$$< j_1 m_1, j_2 = \tfrac{1}{2}, m_2 \,| j_1 j_2; JM >$$

	$m_2 = \frac{1}{2}$	$m_2 = -\frac{1}{2}$
$J = j_1 + \frac{1}{2}$	$\sqrt{\frac{j_1 + m + \frac{1}{2}}{2j_1 + 1}}$	$\sqrt{\frac{j_1 - m + \frac{1}{2}}{2j_1 + 1}}$
$J = j_1 - \frac{1}{2}$	$-\sqrt{\frac{j_1 - m + \frac{1}{2}}{2j_1 + 1}}$	$\sqrt{\frac{j_1 + m + \frac{1}{2}}{2j_1 + 1}}$

$$< j_1 m_1, j_2 = 1 m_2 | j_1 j_2; JM >$$

	$m_2 = 1$	$m_2 = 0$	$m_2 = -1$
$J = j_1 + 1$	$\sqrt{\frac{(j_1 + m)(j_1 + m + 1)}{(2j_1 + 1)(2j_1 + 2)}}$	$\sqrt{\frac{(j_1 - m + 1)(j_1 + m + 1)}{(2j_1 + 1)(j_1 + 1)}}$	$\sqrt{\frac{(j_1 - m)(j_1 - m + 1)}{(2j_1 + 1)(2j_1 + 2)}}$
$J = j_1$	$-\sqrt{\frac{(j_1 + m)(j_1 - m + 1)}{2j_1(j_1 + 1)}}$	$\frac{m}{\sqrt{j_1(j_1 + 1)}}$	$\sqrt{\frac{(j_1 - m)(j_1 + m + 1)}{2j_1(j_1 + 1)}}$
$J = j_1 - 1$	$\sqrt{\frac{(j_1 - m)(j_1 - m + 1)}{2j_1(2j_1 + 1)}}$	$-\sqrt{\frac{(j_1 - m)(j_1 + m)}{j_1(2j_1 + 1)}}$	$\sqrt{\frac{(j_1 + m)(j_1 + m + 1)}{2j_1(2j_1 + 1)}}$