Notes for Lectures on Quantum Mechanics *

JM States Using Ladder Operators

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Explicit values of the Clebsch Gordon coefficients can be obtained by following the procedure outlined below. This is not the only procedure available to get the values of the Clebsch Gordon coefficients. In the solved problem below we show, by means of an example of adding angular momenta $j_1 = 1$ and $j_2 = 1/2$, how the Clebsch Gordon coefficients could be obtained using the properties of angular momentum ladder operators and orthogonality of states with different values of J.

Problem 1: Enumerate all possible states with definite JM values for the case $j_1 = 1, j_2 = \frac{1}{2}$. Using properties of J_- and orthogonality obtain different states $|JM\rangle$ in terms of the states $|j_1 j_2; m_1 m_2\rangle$.

We record the action of ladder operators on states $|jm\rangle$. This will be used repeatedly.

$$J_+|\mathbf{jm}\rangle = \sqrt{j(j+1)-m(m+1)}\,\hbar\,|jm+1\rangle \tag{1}$$
$$J_-|\mathbf{jm}\rangle = \sqrt{j(j+1)-m(m-1)}\,|jm-1\rangle \tag{2}$$

Let $|1m_1;\frac{1}{2}m_2\rangle$ denote the state $|1m_1\rangle|\frac{1}{2}m_2\rangle$ eigen state of $J_z^{(1)}$ and $J_z^{(2)}$ and $|\mathbf{JM}\rangle$ $\{JM$ in bold$\}$ will be used to denote the states with definite values of total angular momentum.

First we take up the construction of the states $|\mathbf{JM}\rangle$ with $J = \frac{3}{2}$. the highest value of total angular momentum.

The states $J = \frac{3}{2}, M = \frac{3}{2}, \frac{1}{2}, -\frac{3}{2}, -\frac{1}{2}$

1. Since $j_1 = 1$ and $j_2 = \frac{1}{2}$ the only possible values of J are $\frac{3}{2}, \frac{1}{2}$.

2. We begin with the highest allowed values of J, M. The state, $|\mathbf{\frac{3}{2}\frac{3}{2}}\rangle$, with highest values for $J = \frac{3}{2}, M = \frac{3}{2}$, corresponds to only one possible state with $m_1 = 1, m_2 = \frac{1}{2}$, the highest value of both m_1 and m_2. Thus

$$|\mathbf{\tfrac{3}{2}\tfrac{3}{2}}\rangle = |1, 1; \tfrac{1}{2}, \tfrac{1}{2}\rangle \tag{3}$$

* qm-lec-17005 Updated:Oct 21, 2021; Ver 0.x

$\boxed{3}$ Next apply $J_- = J_-^{(1)} + J_-^{(2)}$ on Eq.(3) and use (2) to get

$$\sqrt{\tfrac{3}{2}\cdot\tfrac{5}{2} - \tfrac{3}{2}\cdot\tfrac{1}{2}}|\mathbf{\tfrac{3}{2}\tfrac{1}{2}}\rangle = \sqrt{(1.2 - 1.0)}|1,0;\tfrac{1}{2},\tfrac{1}{2}\rangle + \sqrt{\left(\tfrac{1}{2}\cdot\tfrac{3}{2} - \tfrac{1}{2}\cdot(-\tfrac{1}{2})\right)}|1,1;\tfrac{1}{2},-\tfrac{1}{2}\rangle \tag{4}$$

We thus arrive at

$$|\mathbf{\tfrac{3}{2}\tfrac{1}{2}}\rangle = \sqrt{\tfrac{2}{3}}|1,0;\tfrac{1}{2},\tfrac{1}{2}\rangle + \sqrt{\tfrac{1}{3}}|1,1;\tfrac{1}{2},-\tfrac{1}{2}\rangle \tag{5}$$

$\boxed{4}$ Applying $J_- = J_-^{(1)} + J_-^{(2)}$ on Eq.(5) again we get the state with total $M = -\frac{1}{2}$.

$$\begin{aligned} J_-|\mathbf{\tfrac{3}{2}\tfrac{1}{2}}\rangle &= \left(J_-^{(1)} + J_-^{(2)}\right)\left[\sqrt{\tfrac{2}{3}}|1,0;\tfrac{1}{2},\tfrac{1}{2}\rangle + \sqrt{\tfrac{1}{3}}|1,1;\tfrac{1}{2},-\tfrac{1}{2}\rangle\right. \\ &= J_-^{(1)}\left[\sqrt{\tfrac{2}{3}}|1,0;\tfrac{1}{2},\tfrac{1}{2}\rangle + \sqrt{\tfrac{1}{3}}|1,1;\tfrac{1}{2},-\tfrac{1}{2}\rangle\right] \\ &\quad + J_-^{(2)}\left[\sqrt{\tfrac{2}{3}}|1,0;\tfrac{1}{2};\tfrac{1}{2}\rangle + \sqrt{\tfrac{1}{3}}|1,1;\tfrac{1}{2},-\tfrac{1}{2}\rangle\right] \end{aligned} \tag{6}$$

Using (2), we get

$$\begin{aligned} \sqrt{\tfrac{3}{2}\cdot\tfrac{5}{2} - \tfrac{1}{2}\cdot(-\tfrac{1}{2})}|\mathbf{\tfrac{3}{2}}\ -\mathbf{\tfrac{1}{2}}\rangle &= \sqrt{\tfrac{2}{3}}\sqrt{1.2-0}|1,-1;\tfrac{1}{2},\tfrac{1}{2}\rangle + \sqrt{\tfrac{1}{3}}\sqrt{1.2-0}|1,0;\tfrac{1}{2},-\tfrac{1}{2}\rangle \\ &\quad + \sqrt{\tfrac{2}{3}}\sqrt{\tfrac{1}{2}\cdot\tfrac{3}{2} - \tfrac{1}{2}\cdot(-\tfrac{1}{2})}|1,0;\tfrac{1}{2},-\tfrac{1}{2}\rangle + 0 \end{aligned} \tag{7}$$

where the last term in Eq.(6) vanishes because the minimum value of m_2 is $-\frac{1}{2}$. On simplifying we get

$$|\mathbf{\tfrac{3}{2}},-\mathbf{\tfrac{1}{2}}\rangle = \sqrt{\tfrac{1}{3}}|1,-1;\tfrac{1}{2},\tfrac{1}{2}\rangle + \sqrt{\tfrac{2}{3}}|1,0;\tfrac{1}{2},-\tfrac{1}{2}\rangle \tag{8}$$

$\boxed{5}$ Repeating the above procedure of applying J_- we would get

$$|\mathbf{\tfrac{3}{2}},-\mathbf{\tfrac{3}{2}}\rangle = |1,-1;\tfrac{1}{2} - \tfrac{1}{2}\rangle \tag{9}$$

The above relation can also be written down directly because for the state with $J = M = -\frac{3}{2}$ would involve only one combination of m_1, m_2, viz., $m_1 = -1, m_2 = -\frac{1}{2}$ and hence by normalization we must have Eq.(9).

The states $J = \frac{1}{2}, M = \pm\frac{1}{2}$

$\boxed{1}$ We now first consider the state $M = \frac{1}{2}$ for $J = \frac{1}{2}$. The state $|\frac{1}{2}\frac{1}{2}\rangle$ would be linear combination of the states with (m_1, m_2) values given by $(0, \frac{1}{2})$ and $(-1, \frac{1}{2})$ and hence

$$|\mathbf{\tfrac{1}{2}\tfrac{1}{2}}\rangle = \alpha|1,0;\tfrac{1}{2},\tfrac{1}{2}\rangle + \beta|1,1;\tfrac{1}{2},-\tfrac{1}{2}\rangle \tag{10}$$

The same values of (m_1, m_2) also appear in Eq.(5) for the $J = \frac{3}{2}$. The two states in Eq.(5) and Eq.(10) should be orthogonal. The orthogonality gives

$$\sqrt{\tfrac{2}{3}}\alpha + \sqrt{\tfrac{1}{3}}\beta = 0. \tag{11}$$

This equation together with normalization condition $\alpha^2 + \beta^2 = 1$ determines α and β and we get

$$|\mathbf{\tfrac{1}{2}\tfrac{1}{2}}\rangle = \sqrt{\tfrac{1}{3}}|1,0;\tfrac{1}{2}\tfrac{1}{2}\rangle - \sqrt{\tfrac{2}{3}}|1,1;\tfrac{1}{2},-\tfrac{1}{2}\rangle \tag{12}$$

This state is determined up to a phase factor, which has to be fixed by some convention.

[2] Applying J_- on Eq.(12) gives the expression for the state $|\frac{1}{2}, -\frac{1}{2}\rangle$

$$|\mathbf{\tfrac{1}{2}}, -\mathbf{\tfrac{1}{2}}\rangle = \sqrt{\tfrac{2}{3}}|1, -1; \tfrac{1}{2}, \tfrac{1}{2}\rangle - \sqrt{\tfrac{1}{3}}|1, 0; \tfrac{1}{2}, -\tfrac{1}{2}\rangle \tag{13}$$

It is easily verified that state is orthogonal to Eq.(8).

This completes construction of all states $|\mathbf{JM}\rangle$.