QM-17 Lecture Notes

Recurrence Relations for CG Coefficients

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Using ladder operators

In this connection with addition of angular momenta, the following results from the theory of angular momentum derived earlier will be useful.

$$
\begin{aligned}
J_\pm|JM\rangle &= \sqrt{(J(J+1) - M(M\pm 1)}|JM\pm 1\rangle && (1)\\
(J^{(1)}_\pm + J^{(2)}_\pm)|j_1m_1j_2m_2\rangle &= \sqrt{(j_1(j_1+1) - (m_1\pm 1)}|j_1m_1\pm 1j_2m_2\rangle \\
&+ \sqrt{(j_2(j_2+1) - m_2(m_2\pm 1))}|j_1m_1j_2m_2\pm 1\rangle && (2)
\end{aligned}
$$

On taking conjugate of Eq.(2) we get

$$
\begin{aligned}
\langle j_1m_1j_2m_2|(J^{(1)}_\mp + J^{(2)}_\mp) &= \langle j_1(m_1\pm 1)j_2m_2|\sqrt{(j_1(j_1+1) - (m_1\pm 1)} \\
&\quad +\langle j_1m_1j_2(m_2\pm 1)|\sqrt{j_2(j_2+1) - m_2(m_2\pm 1))} && (3)
\end{aligned}
$$

which is a consequence of the angular momentum commutation relations. Considering the matrix element

$$
\langle j_1j_2m_1m_2|J_\pm|JM\rangle = \langle j_1j_2m_1m_2|(J^{(1)}_\pm + J^{(2)}_\pm)|JM\rangle \qquad (4)
$$

and using Eq.(1) and Eq.(3) we get two relations, one for J_+ and

$$
\begin{aligned}
&\sqrt{J(J+1) - M(M+1)}\langle j_1j_2m_1m_2|J(M+1)\rangle \\
&= \langle j_1j_2m_1 - 1m_2|JM\rangle\sqrt{j_1(j_1+1) - m_1(m_1+1)} \\
&\quad +\langle j_1j_2m_1m_2 - 1|JM\rangle\sqrt{j_2(j_2+1) - m_2(m_2+1)} && (5)
\end{aligned}
$$

and a second relation for J_-

$$
\begin{aligned}
&\sqrt{J(J+1) - M(M-1)}\langle j_1m_1, j_2m_2|J(M-1)\rangle \\
&= \langle j_1(m_1+1)j_2m_2|JM\rangle\sqrt{j_1(j_1+1) - m_1(m_1-1)} \\
&\quad +\langle j_1m_1, j_2(m_2+1)|JM\rangle\sqrt{j_2(j_2+1) - m_2(m_2-1)} && (6)
\end{aligned}
$$

We will make repeated use of the results Eq.(5),Eq.(6) given above. These equations can be used successively with $M = J, J-1, \ldots$ to compute the Clebsch Gordon coefficients.

Restrictions on total JM values

The restrictions $1) - 4)$, given above, on the allowed values of the total angular momentum J will be derived by considering the matrix elements $\langle j_1 j_2 m_1 m_2 | J_z | JM\rangle$,$\langle j_1 j_2 m_1 m_2 | J_z | JM\rangle$ and $\langle j_1 j_2 m_1 m_2 | J_\pm | JM\rangle$ and by repeated use of (5) and (6)

<u>**Proof of $M = m_1 + m_2$**</u> The first result is easy to prove. Since $J_z = J_z^{(1)} + J_z^{(2)}$, taking the matrix element and using the properties

$$J_z|JM\rangle = M\hbar|JM\rangle \tag{7}$$

$$(J_z^{(1)} + J_z^{(2)})|j_1 m_1 j_2 m_2\rangle = (m_1 + m_2)\hbar|j_1 m_1 j_2 m_2\rangle \tag{8}$$

we obtain

$$\langle JM|(J_z^{(1)} + J_z^{(2)} - J_z)|j_1 m_1 j_2 m_2\rangle = 0 \tag{9}$$

Therefore,

$$(m_1 + m_2 - M)\langle JM|j_1 m_1 j_2 m_2\rangle = 0 \tag{10}$$

Thus if $M \neq m_1 + m_2$, the Clebsch Gordon coefficient $\langle JM|j_1 j_2 m_1 m_2\rangle$ has to be zero. In other words, a nonzero value of $\langle JM|j_1 j_2 m_1 m_2\rangle$ is possible only when

$$M = m_1 + m_2 \tag{11}$$

<u>**Range of J values**</u> The results will be derived by considering the matrix elements $\langle j_1 m_1 j_2 m_2 | J_z | JM\rangle$ and $\langle j_1 m_1, j_2 m_2 | J_\pm | JM\rangle$.

Note that there is one relation between three the variables $m_1 m_2, M$. Hence we need the Clebsch Gordon coefficients for all allowed values of M and $m_1$which vary in the range $M = -J, \cdots, J$ and $m_1 = -j_1, \cdots, j_1$. We will now argue that these can all be related to single coefficient $\langle JJ|; j_1 J - j_1\rangle$. They can all be related to $\langle j_1 j_1, j_2(J - j_1)|JJ\rangle$

<u>**Use (5) with $M = J, m_1 = j_1$**</u>

$$\begin{aligned} & 0 \times \langle j_1 m_1, j - 2 m_2 | J(M+1)\rangle \\ &= \langle j_1(j_1 - 1), j_2 m_2 | JJ\rangle \sqrt{2j_1} \\ &\quad + \langle j_1 j_1, j_2(m_2 - 1)|JM\rangle \sqrt{j_2(j_2+1) - m_2(m_2 - 1)} \end{aligned} \tag{12}$$

<u>**Use (6) with $M = J, m_1 = j_i$**</u>

$$\begin{aligned} & \sqrt{2J}\langle ; j_1 m_2 | J(J-1)\rangle \\ &= \langle j_1(j_1 + 1), j_2 m_2 | JM\rangle \times 0 \\ &\quad + \langle j_1 j_1, j_2(m_2 + 1)|JM\rangle \sqrt{j_2(j_2+1) - m_2(m_2 + 1)} \end{aligned} \tag{13}$$

<u>**Use (5) with $M = J - 1, m_1 = j_1$**</u>

$$\begin{aligned} & \sqrt{2J}\langle ; j_1 m_2 | JJ\rangle \\ &= \langle ; (j_1 - 1) m_2 | J(J-1)\rangle \sqrt{2j_1} \\ &\quad + \langle j_1 j_1, j_2(m_2 - 1)|J(J-1)\rangle \sqrt{(j_2(j_2+1) - m_2(m_2 - 1)} \end{aligned} \tag{14}$$

Use (6) **with** $M = J, m_1 = j_1 - 1$

$$\begin{aligned}
&\sqrt{2J}\langle j_1(j_1-1), j_2m_2|J(J-1)\rangle \\
&= \langle j_1j_1, j_2m_2|JJ\rangle \times 0 \\
&\quad +\langle j_1(j_1-1), j_2m_2+1|JM\rangle\sqrt{j_2(j_2+1)-m_2(m_2+1)}
\end{aligned} \tag{15}$$

Use (6) **with** $M = J-1, m_1 = j_1$

$$\begin{aligned}
&\sqrt{(J+M)(J-M+1)}\langle j_1j_2m_1m_2|J(J-2)\rangle \\
&= \langle j_1(j_1+1), j_2m_2|JM\rangle\sqrt{j_1(j_1+1)-m_1(m_1+1)} \\
&\quad +\langle j_1j_2j_1(m_2+1)|JM\rangle\sqrt{(j_2(j_2+1)-m_2(m_2+1)}
\end{aligned} \tag{16}$$

We can continue in this fashion. We see that the Clebsch Gordon coefficient for different pairs of values of M, m_1 are known in terms of a single coefficient for $M = J, m_1 = j_i$ as follows

Equation	New Coefficient	Known in terms of
Eq.(12)	$m_1 = j_1 - 1, M = J$	$m_1 = j_1, M = J$
Eq.(13)	$m_1 = j_1, M = J - 1$	$m_1 = j_1, M = J$
Eq.(14)	$m_1 = j_1 - 1, M = J - 1$	$j_1, M = J - 1$ and $m_1 = j_1 - 1, M = J$
Eq.(15)	$m_1 = j_1 - 1, M = J - 1$	$m_1 = j_1 - 1, M = J$
Eq.(16)	$m_1 = j_1, M = J - 2$	$m = j_1, M = J - 1$

Next we consider the state $|j_1m_1, j_2m_2\rangle$ with $m_1 = j_1$ and $m_2 = j_2$. In this state M has the highest value $j_1 + j_2$.

All these coefficients can be fixed in terms of a single coefficient $\langle j_1j_1, j_2m_2 = (j - j_1)|JM\rangle$ which is non zero only if $j - j_1$ lies in between $-j_1$ and j_1. Thus

$$-j_2 \leq j - j_1 \leq j_2 \tag{17}$$

By repeating the above steps with j_1 and j_2 interchanged we would get

$$-j_1 \leq j - j_2 \leq j_1 \tag{18}$$

These two conditions, Eq.(17) and Eq.(18), are equivalent to the requirement $J > |j_1 - j_2|$. Since maximum possible value of $M = m_1 + m_2$ is $j_1 + j_2$ we must have $J < j_1 + j_2$. Thus the total angular momentum is constrained to lie between $|j_1 - j_2|$ and $j_1 + j_2$. The three numbers J, j_1, j_2 should be such that they satisfy triangle inequalities

$$|j_1 - j_2| \leq J \leq j_1 + j_2. \tag{19}$$

It should be remarked that Eq.(17)-Eq.(19), written as conditions on j_1 and j_2, are equivalent to each of the following two alternate forms

$$|J - j_2| \leq j_1 \leq J + j_2 \qquad \text{and} \qquad |J - j_1| \leq j_2 \leq J + j_1. \tag{20}$$

J takes values in steps of 1 The range of J, the total angular momentum value, has been determined; the minimum value being $|j_1 - j_2|$ and the maximum value is $(j_1 + j_2)$. Are all integral, and half integral values, allowed in this range allowed ? We must fix which values of J are allowed and which ones are not allowed?

The state $|j_1 j_2 m_1 m_2\rangle$, when both m_1 and m_2 have their maximum allowed values j_1 and j_2, the value of $M = m_1 + m_2$ is maximum and equal to $j_1 + j_2$. There is only one such state and this must correspond to $J = j_1 + j_2$.

$$|J = (j_1 + j_2), M = (j_1 + j_2)\rangle = |j_1 m_1 = j_1, j_2 m_2 = j_2\rangle \tag{21}$$

The next value of $M = j_1 + j_2 - 1$ corresponds to two linearly independent states corresponding to

(a) $m_1 = j_1 - 1, m_2 = j_2$, and

(b) $m_1 = j_1, m_2 = j_2 - 1$

What are the corresponding J values? One combination of these two states must be the $M = j_1 + j_2 - 1$ partner of the state Eq.(21); and the other linear combination can only correspond to the next value $J = j_1 + j_2 - 1$.Continuing in this way, we now consider $M = j_1 + j_2 - 2$ which will come from three sets of m_1, m_2 values,*i.e.*,

(a) $m_1 = j_1 - 2, m_2 = j_2$

(b) $m_1 = j_1 - 1, m_2 = j_2 - 1$, and,

(c) $m_1 = j_1, m_2 = j_2 - 2$

Of the three states $|j_1 m_1, j_2 m_2\rangle$ corresponding the above values two linear combinations will correspond to the J values $j_1 + j_2$ already found; a third linear combination must therefore correspond to the value $J = j_1 + j_2 - 2$. Proceeding in this fashion we see that the successive J values differ by one. *How are we sure that all the J values have been correctly identified ?*. We will now count the number of states in two different ways to confirm the conclusion that J takes all the values in the allowed range from $|j_1 - j_2|$ to $j_1 + j_2$.

Count in two ways to cross check Thus we have two set of orthonormal bases

$$\Big\{|j_1 m_1, j_2 m_2\rangle \Big| m_1 = -j_1, \cdots, j_1, m_2 = -j_2, j_2 \cdots \Big\} \tag{22}$$

and

$$\Big\{|j_1 j_2; JM\rangle \Big| J = |j_1 - j_2|, \cdots, j_1 + j_2, M = -J, \cdots, J\Big\} \tag{23}$$

It is easily seen that the total number of vectors in two bases are equal. The number of elements in the set (22)) is $(2j_1+1)(2j_2+1)$ and in the set (23) is also the same

$$\sum_{J=|j_1-j_2|}^{j_1+j_2} (2J+1) = (2j_1+1)(2j_2+1). \tag{24}$$

The two bases in Eq.(22) and Eq.(23) are orthonormal and hence the transformation connecting the two must be a unitary transformation and the Clebsch Gordon coefficients must satisfy the relations

$$\sum_{JM} \langle j_1 j_2; m_1 m_2 | j_1 j_2 JM\rangle \langle JM | j_1 j_2; m_1' m_2'\rangle = \delta_{m_1 m_1'} \delta_{m_2 m_2'} \tag{25}$$

$$\sum_{m_1, m_2} \langle JM | j_1 j_2; m_1 m_2\rangle \langle j_1 j_2; m_1 m_2 | j_1 j_2 J' M'\rangle = \delta_{JJ'} \delta_{MM'} \tag{26}$$

These relations can also be seen as a consequence of the completeness formula such as

$$\sum_{JM} |JM\rangle\langle JM| = \hat{I} \tag{27}$$

and a similar relation for the vectors $|j_1 j_2; m_1 m_2\rangle$.

The Clebsch Gordon coefficients for addition of angular momenta are tabulated. Two such tables for $j_2 = \frac{1}{2}$ and $j_2 = 1$ are given on the next page.

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