Notes for Lectures on Quantum Mechanics
Addition of Angular Momenta — Statement of Problem

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Consider two sets of angular momentum operators $\vec{J}^{(1)}$ and $\vec{J}^{(2)}$

$$[J_x^{(\alpha)}, J_y^{(\alpha)}] = i\hbar J_z^{(\alpha)}, \qquad \alpha = 1, 2 \tag{1}$$

such that every component of $\vec{J}^{(1)}$ commutes with every component of $\vec{J}^{(2)}$. We write

$$[\vec{J}^{(1)}, \vec{J}^{(2)}] = 0 \ . \tag{2}$$

It is obvious that $\vec{J}^{(1)2}, \vec{J}^{(2)2}, J_z^{(1)}$ and $J_z^{(2)}$ from a commuting set. Let us now define total angular momentum $\vec{J} = \vec{J}^{(1)} + \vec{J}^{(2)}$ then

(a) $\vec{J}$ satisfies angular momentum commutation relations

$$[J_x, J_y] = i\hbar J_z \qquad \text{etc.} \tag{3}$$

(b) All the three components of $\vec{J}$ commute with $\vec{J}_1^2$ and $\vec{J}_2^2$.

(c) It follows from (b) that $\vec{J}^2$ commutes with $\vec{J}_1^2$ and $\vec{J}_2^2$.

The angular momentum commutation relations for $\vec{J}$ imply that the eigenvalues of J^2 and J_z are the same as already obtained. In fact the component of $\vec{J}$ along every unit vector $\hat{n}$ has the same eigenvalues as J_z which are given by

Angular momentum eigenvalues

Operator	Eigenvalues	Remarks
J^2	$J(J+1)\hbar^2$	$J = 0, 1/2, 1, \cdots$
J_z	$M\hbar$	values of M are from $-J$ to J in steps of unity

Given the angular momentum commutation relations it is straight forward to see that one can have two sets of commuting operators. Each of these two sets of operators can have simultaneous eigenvectors. The two commuting sets of operators and the notation for their simultaneous eigenvectors is shown in the table below.

Commuting Set	Simultaneous Eigenvectors
$J^{(1)^2}, J^{(2)^2}, J_z^{(1)} J_z^{(2)}$	$|j_1 m_1, j_2 m_2\rangle$
$J^{(1)^2}, J^{(2)^2}, J^2, J_z$	$|j_1, j_2; JM\rangle$

The vectors $|j_1 m_1, j_2 m_2\rangle$ listed in the second column of the first row are just the direct product of vectors $|j_1 m_1\rangle$ and $|j_2 m_2\rangle$ which are eigenvectors of $\vec{J}^{(1)2}, J_z^{(1)}$ and of $\vec{J}^{(2)2}, J_z^{(2)}$, respectively.

$$|j_1 m_1, j_2 m_2\rangle \equiv |j_1 m_1\rangle \otimes |j_2 m_2\rangle \tag{4}$$

The problem of addition of angular momenta consists of following two questions:

(i) For a state $|j_1 m_1, j_2 m_2\rangle$, which has definite values j_1, m_1 and j_2, m_2 for the individual values of the square, and the z- component of the angular momenta, what values of square of total angular momentum $\vec{J}^2$ and J_z are allowed?

(ii) The second part of the problem is to construct the transformation matrix connecting the two sets of simultaneous eigenvectors, one set for each commuting set.

The states $|j_1 j_2; JM\rangle$ can be expressed as linear combination of states $|j_1 m_1, j_2 m_2\rangle$ and vice versa

$$|j_1 m_1, j_2 m_2\rangle = \sum_{JM} C(JM; j_1 j_2 m_1 m_2)|j_1 j_2; JM\rangle \tag{5}$$

The coefficients $C(JM; j_1 j_2 m_1 m_2)$, appearing n the above equation, are simply the scalar products

$$C(JM; j_1 j_2 m_1 m_2) = \langle j_1 j_2, JM|j_1 j_2, m_1 m_2\rangle \tag{6}$$

where $M = m_1 + m_2$.

Conversely the states with definite total angular momenta are linear combinations of the product states $|j_1 m_1, j_2 m_2\rangle$:

$$|j_1 j_2; JM\rangle = \sum_{\substack{m_1, m_2 \\ m_1 + m_2 = M}} \langle j_1 j_2, m_1 m_2|j_1 j_2, JM\rangle |j_1 m_1, j_2 m_2\rangle. \tag{7}$$

Since the values of j_1, j_2 are the common for all the states, we shall frequently suppress these labels and write the ket $|j_1 j_2; JM\rangle$ as $|JM\rangle$. The coefficients $\langle j_1 j_2, JM|j_1 m_1, j_2 m_2\rangle$, written in short as $\langle JM|j_1 m_1, j_2 m_2\rangle$, are known as Clebsch Gordon coefficients and expressions are tabulated in books. The problem of addition of angular momenta consists of finding allowed range of values of J, M and obtaining expressions for the Clebsch Gordon coefficients.