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Angular Momentum in Quantum Mechanics — Summary of results

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1. The angular momentum commutation relations are

$$[J_x, J_y] = i\hbar J_z; \tag{1}$$
$$[J_y, J_z] = i\hbar J_x; \tag{2}$$
$$[J_z, J_x] = i\hbar J_y. \tag{3}$$

2. It is useful to introduce the operators $J_\pm = J_x \pm iJ_y$ and these operators obey commutation relations

$$[J_+, J_-] = 2\hbar J_z \tag{4}$$
$$[J_z, J_+] = \hbar J_+ \tag{5}$$
$$[J_z, J_-] = -\hbar J_- \tag{6}$$

   Also $\vec{J}^2$ commute with all the three components of the angular momentum J_x, J_y, J_z.

3. Some useful relations are

$$J_+J_- = J^2 - J_z^2 + \hbar J_z \tag{7}$$
$$J_-J_+ = J^2 - J_z^2 - \hbar J_z \tag{8}$$
$$J^2 = \frac{1}{2}(J_+J_- + J_-J_+) + J_z^2 \ . \tag{9}$$

4. One can find simultaneous eigenvectors of $\vec{J}^2$ and *any one* component, $J_{\hat{n}} = \hat{n}\cdot\vec{J}$, along a fixed direction given by the unit vector $\hat{n}$.

5. The eigenvalues of $\vec{J}^2$ are $j(j+1)\hbar^2$ where j can be integer of half integer. The eigenvalues of a component of J_n take values from $-j$ to j in step of 1.

6. It is customary to denote the *normalized*, simultaneous, eigenvectors of $\vec{J}^2$ and J_z by $|jm\rangle$ so that

$$\vec{J}^2|jm\rangle = j(j+1)\hbar|jm\rangle \tag{10}$$
$$J_z|jm\rangle = m\hbar|jm\rangle \tag{11}$$

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7. The operators $J_\pm$ acting on $|jm\rangle$ give a ket vector *proportional* to $|jm \pm 1\rangle$. The proportionality coefficient can be worked out using the relation Eq.(7) and Eq.(8). One then gets

$$J_\pm|jm\rangle = \sqrt{j(j+1) - m(m\pm 1)}\,\hbar\,|jm\pm 1\rangle \tag{12}$$

8. The operators $J_\pm$ annihilate $|j, \pm j\rangle$ because the J_z value cannot be increased beyond j nor can it be decreases below $-j$.

$$J_+|j, j\rangle = 0 \qquad\qquad J_-|j, -j\rangle = 0 \tag{13}$$

9. The above results are applicable to operators satisfying angular momentum commutation relations except that the half integral values are ruled out for the orbital angular momentum, because of additional requirement of single valuedness of the wave function.