Notes For Lectures on Quantum Mechanics *

Classical Motion in Three Dimensions

Spherically symmetric potentials

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Abstract

Under influence of a spherically symmetric potential, $V(r)$ the orbit of a particle is confined to a plane as a consequence of angular momentum conservation. The radial motion is like motion in one dimension in an effective potential

$$V_{\text{eff}}(r) = V(r) + \frac{L^2}{2mr^2}$$

where L is the total angular momentum.

We will briefly recall some results from classical mechanics on motion in spherically symmetric potential problems in three dimensions.

The Lagrangian for a particle in three dimensions is

$$L = \frac{1}{2}m\dot{\vec{r}}^2 - V(\vec{r}) \tag{1}$$

The corresponding Hamiltonian is

$$H = \frac{\vec{p}^2}{2m} + V(\vec{r}) \tag{2}$$

A potential is spherically symmetric if it depends in r alone; it does not depend on θ, ϕ. All the three components of angular momentum, $\vec{L} = \vec{r} \times \vec{p}$, are constants of motion.

For spherically symmetric potentials, it turns out to be useful to work in spherical polar coordinates. In polar coordinates, (r, θ, ϕ) the Lagrangian takes the form

$$L = \frac{1}{2}m\big(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\big) - V(r) \tag{3}$$

The Hamiltonian in polar coordinates is

$$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{p_\phi^2}{2mr^2\sin^2\theta} + V(r) \tag{4}$$

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Properties of classical motion As a consequence of angular momentum conservation the motion of the particle is confined to a plane. Taking this plane to be $X - Y$ plane, one can use plane polar coordinates r, θ to describe the motion. The corresponding Hamiltonian is

$$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + V(r) \tag{5}$$

Here θ is seen to be a cyclic coordinate and corresponding momentum, p_θ, is a constant of motion. p_θ can be seen to be the total angular momentum. The radial motion is then completely described by the effective potential

$$V_{\text{eff}} = V(r) + \frac{L^2}{2mr^2} \tag{6}$$

Here L denotes total angular momentum p_θ.

In quantum mechanics all the features of motion in spherically symmetric potential, except one, described above remain true. In quantum mechanics, the orbits are not well defined and correspondence with the motion being confined to a plane can be established only in the limit of large angular momentum.