Notes for Lectures on Quantum Mechanics *

Free Particle Solution in Polar Coordinates

A. K. Kapoor
http://0space.org/users/kapoor
akkapoor@cmi.ac.in; akkhcu@gmail.com

The radial equation for a free particle, $V(r) = 0$, for all r is

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \left(k^2 - \frac{\ell(\ell+1)}{r^2}\right)R = 0, \tag{1}$$

where $k^2 = \dfrac{2mE}{\hbar^2}$. The solution of the radial equation has the most general form

$$R(r) = Aj_\ell(kr) + Bn_\ell(kr) \tag{2}$$

but we must set $B = 0$ because $n_\ell(kr) \to \infty$ as $r \to 0$. Hence we get

$$R_\ell(r) = Aj_\ell(kr) \tag{3}$$

and the full free particle wave function is

$$\Psi(r,\theta,\phi) = Nj_\ell(kr)Y_{\ell m}(\theta,\phi) \tag{4}$$

For a given value of energy E, ℓ can take all values $0, 1, 2, \ldots.$ and m has $(2\ell+1)$ values from $-\ell$ to ℓ. Therefore, for every energy value $E > 0$ there are infinite number of solutions. If we take linear combinations of solutions with fixed energy E we get most general form of the solution for a given energy as

$$\Phi(\vec{r}) = \sum C_{\ell m} j_\ell(kr)Y_{\ell m}(\theta,\phi). \tag{5}$$

In Cartesian coordinates the free particle solutions for energy E are plane waves $\exp(i\vec{k}\cdot\vec{r})$. Thus it is possible to write each of these two type of solutions as a linear combination of the other type. In particular we have

$$\exp(\vec{k}\cdot\vec{r}) = \sum C_{\ell m} j_\ell(kr)Y_{\ell m}(\theta,\phi) \tag{6}$$

As a particular case of this equation, when k is along the z axis and $\vec{k}\cdot\vec{r} = kz$, we have the expansion of plane waves

$$\exp(ikz) = \sum_0^\infty (2\ell+1)i^\ell j_\ell(kr)P_\ell(\cos\theta) \tag{7}$$

Note that only $m = 0$ terms contribute in the above equation.

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