Notes for Lectures on Quantum Mechanics *
Solution of Radial Equation for a Constant Potential

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1 Piecewise constant potentials

The solutions of the radial equation for a constant potential in three dimensions are known in terms of spherical Bessel functions. We shall list these solutions and discuss their properties before taking specific examples such as free particle, square well potential. Let us assume $V(r) = V_0$for some range of values of r. Then for this range of values the radial equation takes the form

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \left(\frac{2m}{\hbar^2}(E - V_0) - \frac{\ell(\ell+1)}{r^2}\right)R(r) = 0, \quad a < r < b. \tag{1}$$

where a may be zero and b may be infinity. We shall consider the cases $E - V_0 > 0$ and $E - V_0 < 0$ separately.

CASE I: $E - V_0 > 0$

We define $\dfrac{2m(E - V_0)}{\hbar^2} = k^2$ and the two linearly independent solutions of the radial equation are given by $j_\ell(kr)$ and $n_\ell(kr)$ known as spherical Bessel functions. These are related to the Bessel functions $J\_\nu(kr)$ as follows

$$j_\ell(kr) = \left(\frac{\pi}{2kr}\right)^{1/2} J_{\ell+\frac{1}{2}}(kr) \tag{2}$$

$$n_\ell(kr) = \left(\frac{\pi}{2kr}\right)^{1/2} (-1)^{\ell+1} J_{-\ell-\frac{1}{2}}(kr) \tag{3}$$

and the most general solution of the radial equation is a linear combination of the above solutions.

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* qm-lec-16001 Updated:; Ver 0.x

$$R(r) = Aj_\ell(kr) + Bn_\ell(kr) \tag{4}$$

2 Solution near origin and at large distances

We need to know the behaviour of the solutions for $r \approx 0$ and for $r \to \infty$.

Small r **:** The solution $j_\ell(kr)$ goes to zero but $n_\ell(kr)$ is singular for $r \approx 0$.In fact as $\rho \to 0$, we have

$$j_\ell(\rho) \to \frac{\rho^\ell}{(2\ell+1)!!} \tag{5}$$

$$n_\ell(\rho) \to (2\ell-1)!!\rho^{-\ell-1} \tag{6}$$

Large r **:** For large ρ both j_ℓ and n_ℓ are oscillatory. As $\rho \to \infty$

$$j_\ell(\rho) \to \frac{1}{\rho}\cos(\rho - (\ell+1)\pi/2) \tag{7}$$

$$n_\ell(\rho) \to \frac{1}{\rho}\sin(\rho - (\ell+1)\pi/2) \tag{8}$$

Thus for $E > V_0$ both $j_\ell(\rho)$ and $n_\ell(\rho)$ are acceptable solutions as $\rho \to \infty$

CASE II : $E - V_0 < 0$

In this case we define

$$\frac{2m(E-V_0)}{\hbar^2} = -\alpha^2, \qquad \alpha = \text{real} \tag{9}$$

In this case two linearly independent solutions are $j_\ell(i\alpha r)$ and $n_\ell(i\alpha r)$ and the most general solution is

$$R(r) = Aj_\ell(i\alpha r) + Bn_\ell(i\alpha r) \tag{10}$$

Again $n_\ell(i\alpha r)$ has unacceptable singular behaviour at $r = 0$. To discuss large r behaviour, introduce Hankel functions of first and second kinds by

$$h_\ell^{(1)}(\rho) = j_\ell(\rho) + in_\ell(\rho) \tag{11}$$

$$h_\ell^{(2)}(\rho) = j_\ell(\rho) - in_\ell(\rho) \tag{12}$$

Then, as $\rho \to \infty$, we have

$$h_\ell^{(1)}(\rho) \to -\frac{1}{\rho}\exp(-\rho), \text{ and } h_\ell^{(2)}(\rho) \to \frac{1}{\rho}\exp(\rho) \tag{13}$$

and $h_\ell^{(2)}(\rho)$ blows up and becomes infinite as $\rho \to \infty$. and is unacceptable. It may be remarked that the solutions $j_\ell(\rho)$ and $n_\ell(\rho)$ are linear combinations of $\cos\rho, \sin\rho$ multiplied by powers of ρ. Similarly, $h_\ell^{(1,2)}(\rho)$ are exponentials multiplied by terms containing powers of ρ.

Table: Forms of acceptable solutions of radial equation

	Near $r=0$	For large r	for $'r'$ in (a,b)
$E-V_0>0$	$j_\ell(kr)$	$Aj_\ell(kr)+Bn_\ell(kr)$	$Aj_\ell(kr)+Bn_\ell(kr)$
$E-V_0<0$	$j_\ell(ikr)$	$j_\ell(ikr)+in_\ell(ikr)$ $\equiv h_\ell^{(1)}(ikr)$	$Aj_\ell(ikr)+Bn_\ell(ikr)$ $\equiv h_\ell^{(1)}(ikr)$

3 Some Spherical Bessel functions

We shall now tabulate first few spherical Bessel functions.

$$j_0(\rho) = \frac{\sin\rho}{\rho} \tag{14}$$

$$j_1(\rho) = \frac{\sin\rho}{\rho^2} - \frac{\cos\rho}{\rho} \tag{15}$$

$$j_2(\rho) = \left(\frac{3}{\rho^3} - \frac{1}{\rho}\right)\sin\rho - \frac{3}{\rho^2}\cos\rho \tag{16}$$

$$n_0(\rho) = -\frac{\cos\rho}{\rho} \tag{17}$$

$$n_1(\rho) = -\frac{\cos\rho}{\rho^2} - \frac{\sin\rho}{\rho} \tag{18}$$

$$n_2(\rho) = -\left(\frac{3}{\rho^3} - \frac{1}{\rho}\right)\cos\rho - \frac{3}{\rho^2}\sin\rho \tag{19}$$

The first few Hankel functions are given below.

$$h_0^{(1)}(i\rho) = -\frac{1}{\rho}\exp(-\rho) \tag{20}$$

$$h_1^{(1)}(i\rho) = i\left(\frac{1}{\rho} + \frac{1}{\rho^2}\right)\exp(-\rho) \tag{21}$$

$$h_2^{(1)}(i\rho) = \left(\frac{1}{\rho} + \frac{3}{\rho^2} + \frac{3}{\rho^3}\right)\exp(-\rho) \tag{22}$$

The functions $h_\ell^{(2)}(\rho)$ have $\exp(\rho)$ as a factor which is bad for large ρ; only $h^{(1)}(i\rho)$ is useful when one needs a solution valid for large ρ. When one needs a solution valid for intermediate values of ρ, one can take linear combination of $j_\ell(\rho)$ and $n_\ell(\rho)$.