Notes for Lectures on Quantum Mechanics *

Angular Momentum Algebra — Coordinate Representation

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Contents

The orbital angular momentum of a particle is given by $\vec{L} = \vec{r} \times \vec{p}$ and the components of the angular momentum operator in coordinate representation are

$$\hat{L}_x = -i\hbar\Big(\hat{y}\frac{\partial}{\partial z} - \hat{z}\frac{\partial}{\partial y}\Big) \tag{1}$$

$$\hat{L}_y = -i\hbar\Big(\hat{z}\frac{\partial}{\partial x} - \hat{x}\frac{\partial}{\partial z}\Big) \tag{2}$$

$$\hat{L}_z = -i\hbar\Big(\hat{x}\frac{\partial}{\partial y} - \hat{y}\frac{\partial}{\partial x}\Big) \tag{3}$$

Here $\hat{A}$ means operator corresponding to the dynamical variable A. In terms of spherical polar coordinates these expressions take the form

$$\hat{L}_x = i\hbar\left(\sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi}\right) \tag{4}$$

$$\hat{L}_y = i\hbar\left(-\cos\phi\frac{\partial}{\partial\theta} + \cot\theta\sin\phi\frac{\partial}{\partial\phi}\right) \tag{5}$$

$$\hat{L}_z = i\hbar\frac{\partial}{\partial\phi} \tag{6}$$

The operator $\vec{L}^2$ given by

$$\vec{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2 \tag{7}$$

takes the form

$$\vec{L}^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] \tag{8}$$

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The components of orbital angular momentum satisfy commutation relations

$$[L_x, L_y] = i\hbar L_z; \quad [L_y, L_z] = i\hbar L_x; \quad [L_z, L_x] = i\hbar L_y;$$

1 Eigenvalues and Eigenvectors

These commutation relations of angular momentum imply that $\vec{L}^2$ commutes with $\vec{n} \cdot \hat{L}$ for all numerical$\hat{n}$. Hence we can find simultaneous eigenfunctions of $\vec{L}^2$ and a component of $\vec{L}$. along any direction $\vec{n}$. Taking $\hat{n}$ to be along $z-$ axis the eigenvalue equations

$$\vec{L}^2 Y(\theta, \phi) = \lambda \hbar^2 Y(\theta, \phi) \tag{9}$$

$$L_z Y(\theta, \phi) = \mu \hbar Y(\theta, \phi) \tag{10}$$

become differential equations

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] Y(\theta, \phi) + \lambda Y(\theta, \phi) = 0 \tag{11}$$

and

$$-i\frac{\partial}{\partial\phi} Y(\theta, \phi) = \mu Y(\theta, \phi) \tag{12}$$

We shall now show that acceptable solutions exist only for

$$\lambda = \ell(\ell+1); \qquad \mu = m \tag{13}$$

where ℓ can take only positive integral values $0, 1, 2, \cdots$ and m must satisfy the relation $(-\ell \leq m \leq \ell)$, taking values in steps of 1:

$$m = \ell, \ell - 1, \cdots, -\ell + 1, -\ell. \tag{14}$$

There are $(2\ell+1)$ eigenvalues of L_z for a fixed $\vec{L}^2$ and the spherical harmonics $Y_{\ell m}\theta, \phi$ will be seen to be the corresponding eigenfunctions. These results on eigenvalues and eigenfunctions of $\vec{L}^2$ and L_z will be proved by solving the differential equations by the method of separation of variables.

2 Separation of Variables

To solve the differential equations we substitute

$$Y(\theta, \phi) = Q(\theta) E(\phi) \tag{15}$$

in Eq.(11) and (12) and divide by $Y(\theta, \phi) = Q(\theta)E(\phi)$. This gives

$$-i\frac{dE(\phi)}{d\phi} = \mu E(\phi) \tag{16}$$

Similarly, (11) gives

$$\left[\frac{1}{Q(\theta)}\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}Q(\theta)\right) + \frac{1}{\sin^2\theta}\frac{1}{E(\phi)}\frac{d^2E(\phi)}{d\phi^2}\right] + \lambda = 0 \tag{17}$$

On using Eq.(16) in (17) we get

$$\sin^2\theta\left\{\frac{1}{Q(\theta)}\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}Q(\theta)\right)\right\} + \lambda\sin^2\theta = -\frac{1}{E(\phi)}\frac{d^2E(\phi)}{d\phi^2} \tag{18}$$

While the left hand side of the above equation is a function of θ, the right hand side is a function of ϕ alone. Hence each side must be a constant, from Eq.(16) this constant is μ. Thus we get

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}Q(\theta)\right)+\left(\lambda-\frac{\mu^2}{\sin^2\theta}\right)Q(\theta)=0 \tag{19}$$

3 Solution of ϕ equation

General solution of Eq.(16) is

$$E(\phi)=\begin{cases}A\exp(i\sqrt{\mu}\phi)+B\exp(-i\sqrt{\mu}\phi), & \text{if } \mu\neq 0\\ C+D\phi, & \text{if } \mu=0\end{cases} \tag{20}$$

A wave function must be single valued function. For a fixed r,θ,ϕ the values of ϕ and $\phi+2\pi$ correspond to the same point. Hence the solution should have the same value for ϕ and $\phi+2\pi$. Thus we demand that $E(\phi)$ must satisfy

$$E(\phi+2\pi)=E(\phi) \tag{21}$$

for all ϕ. For $\mu=0$ this implies that $D=0$.
Next, when $\mu\neq 0$ we must have

$$A\exp(i\sqrt{\mu}(\phi+2\pi))+B\exp(-i\sqrt{\mu}(\phi+2\pi))=A\exp(i\sqrt{\mu}\phi)+B\exp(-i\sqrt{\mu}\phi) \tag{22}$$

or

$$A\exp(i\sqrt{\mu}\phi)\exp(2\pi i)+B\exp(-i\sqrt{\mu}\phi)\exp(2\pi i))=A\exp(i\sqrt{\mu}\phi)+B\exp(-i\sqrt{\mu}\phi). \tag{23}$$

For $\mu\neq 0$, the linear independence of the $\exp(\pm i\sqrt{\mu}\phi)$ implies that the corresponding coefficients must be separately equal implying that m is an integer. Thus the solutions of Eq.(16) are

$$E(\phi)=\exp(im\phi), \qquad m=0,\pm1,\pm2,\cdots \tag{24}$$

4 Solution of θ equation

If we substitute $w=\cos\theta$ in Eq.(19) takes the form

$$\frac{d}{dw}(1-w^2)\frac{dP(w)}{dw}+\left(\lambda-\frac{m^2}{1-w^2}\right)P(w)=0 \tag{25}$$

where we have introduced $P(w)\equiv Q(\cos\theta)$ and have used

$$\frac{dP(w)}{d\theta}=\frac{dP(w)}{dw}\cdot\frac{dw}{d\theta}=-\sin\theta\frac{dP(w)}{dw}$$

The equation (25) is known as associated Legendre equation. This equation can be solved by the method of series solution. Since (25) is a second order differential equation, there are two linearly independent solutions of this equation. For general values of λ both the solutions become infinite at $w=\pm1$ corresponding to $\theta=0,\pi$ These solutions are therefore unacceptable. For special values $\lambda=\ell(\ell+1)$, where

ℓ is a positive integer, and with $|m| \leq \ell$, one solution remains finite , but not the other solution. Thus we fix

$$\lambda = \ell(\ell+1) \qquad |m| \leq \ell \tag{26}$$

For the above choice, the non singular solution for $P(w)$ is known as the associated Legendre function and has the form

$$P^\ell_m(w) = (1-w^2)^{|m|/2} \frac{d^{|m|}}{dw^{|m|}} P_\ell(w) \tag{27}$$

where $P_\ell(w)$ is Legendre polynomial of degree ℓ.Thus the eigenfunctions of $\vec{L}^2$ and L_z are the

$$Y_{\ell m}(\theta,\phi) = N P^\ell_m(\cos\theta) e^{im\phi}, \qquad\qquad m = \ell, \ell-1, \cdots, \ell \tag{28}$$

The normalization is fixed by demanding

$$\int_0^{2\pi} d\phi \int_0^\pi Y^*_{\ell m}(\theta,\phi) Y_{\ell m}(\theta,\phi) \sin\theta d\theta = 1 \tag{29}$$

The functions $Y_{\ell m}(\theta,\phi)$ in Eq.(28) are known as spherical harmonics.