Optics Mechanics Analogy
Road to Wave Mechanics

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Abstract

We follow Fermi, "Lectures on Quantum Mechanics" to motivate Schrödinger equation.

Point Particle	**Waves**
Mass Point	Wave packet
Trajectory	Ray
Velocity,ϑ	Group Velocity,$\tilde{\vartheta}$
	Phase Velocity, (v)
Potential Energy $\vartheta(x)$	Refractive Index,
	Function of position $n(x)$
Energy E	Frequency ν
	Dispersive medium, $n(,\nu,x), v(\nu,x)$
Manupertuis Principle:	Fermat Priniciple:

$$\int \sqrt{E - \vartheta(x)}\, dx = \min \qquad \int \frac{ds}{v(\nu, x)} = \min$$

Remark Note that,for a particle, $\int \sqrt{E - V(x)}\, dx$ =min, means $\int pdx = \min$ which, for E =const, implies $\int (pdx - Hdt) = \min$.
Particles have dual nature, therefore consistency requires that both Maupertuis and Fermat principles should give the same answer. Therefore, we must have

$$\frac{1}{v(\nu, x)} = f(\nu)\sqrt{E - V(x)} \tag{1}$$

Velocity of a point mass is

$$v = \frac{p}{m} = \sqrt{\frac{2}{m}\Big(E - V(x)\Big)}. \tag{2}$$

On the other hand, the group velocity, for waves, is given by

$$v_g = \frac{d\omega}{dk} = 1/\Big(\frac{dk}{d\omega}\Big) \tag{3}$$

Note that $v = \nu\lambda, \omega = 2\pi\nu$ and $k = 2\pi/\lambda$,

$$v_g = 1\Big/\frac{d}{d\nu}(1/\lambda) = 1\Big/\frac{d}{d\nu}\Big(\frac{\nu}{v(\nu)}\Big) \tag{4}$$

Using (1) the group velocity $\widetilde{\vartheta}$ of waves is given by

$$\frac{1}{\widetilde{\vartheta}} = \frac{d}{d\nu}\Big(\frac{\nu}{v(\nu, x)}\Big) = \frac{d}{d\nu}\Big(\nu f(\nu)\sqrt{E(\nu) - V(x)}\Big). \tag{5}$$

Velocity of a mass point v corresponds to the group velocity v_g of wave packet $(= \widetilde{\vartheta})$. Hence from (2) and (5) we obtain

$$\sqrt{\frac{m}{2}}\frac{1}{\sqrt{E - V(x)}} = \frac{d}{d\nu}\Big(\nu f(\nu)\sqrt{E(\nu) - V(x)}\Big) \tag{6}$$

$$= \frac{d}{d\nu}(\nu f(\nu))\sqrt{E(\nu) - V(x)} + \frac{\nu f(\nu)}{2}\frac{dE(\nu)}{d\nu}\frac{1}{\sqrt{E(\nu) - V(x)}} \tag{7}$$

This equation will be correct for all x and all $\vartheta(x)$ if

$$\frac{d}{d\nu}(\nu f(\nu)) = 0 \text{ or } \nu f(\nu) = \text{const,K}. \tag{8}$$

and the coefficient of $\frac{1}{\sqrt{E - V(x)}}$ on both sides are equal:

$$\sqrt{\frac{m}{2}} = \frac{\nu f(\nu)}{2}\frac{dE(\nu)}{d\nu} = \frac{K}{2}\frac{dE(\nu)}{d\nu} \tag{9}$$

$$\Rightarrow \frac{dE(\nu)}{d\nu} = \text{const}, h \Rightarrow E = h\nu + \text{const.}, \tag{10}$$

Setting this last constant to zero gives

$$\boxed{E = h\nu} \tag{11}$$

and Eq.(10) determines the constant $K = \frac{\sqrt{(2m)}}{h}$. Also $\nu f(\nu) = K$ along with Eq.(1)implies

$$\frac{1}{v(\nu, x)} = \frac{K}{\nu}\sqrt{E(\nu) - V(x)} = \frac{\sqrt{2m}}{h\nu}\sqrt{E(\nu) - V(x)} \tag{12}$$

or

$$\frac{h}{\sqrt{2m(E(\nu) - V(x))}} = \frac{v}{\nu} = \lambda \tag{13}$$

This gives

$$\boxed{\lambda = \frac{h}{p}} \tag{14}$$

Derivation of Schrödinger Equation For monochromatic waves

$$\nabla^2 \psi - \frac{1}{v^2} \frac{d^2 \psi}{dt^2} = 0 \tag{15}$$

We set $\psi(x, t) = u(x) e^{-i\omega t}$, assuming ω to be fixed, using Eq. (13) we get

$$\nabla^2 u + \frac{\omega^2}{v^2} u = 0 \tag{16}$$

$$\Rightarrow \nabla^2 u + \frac{4\pi^2 \nu^2}{h^2 \nu^2} (2m(E(\nu) - V(x))u = 0 \tag{17}$$

$$\Rightarrow \nabla^2 u + \frac{2m}{\hbar^2} (E(\nu) - V(x))u = 0 \tag{18}$$

For states with fixed energy, we have

$$E\psi = h\omega\psi \sim i\hbar \frac{\partial \psi}{\partial t}, \tag{19}$$

giving the time dependent Schrödinger equation

$$ih \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + V(x)\psi \tag{20}$$

Clap!Clap!

Remarks:

The highlight of this route is the derivation of de Broglie relation $\lambda = h/p$ and the Einstein relation $E = h\nu$ for point particles by demanding the action principles for matter and waves give the same result.

I thank Bindu Bambah for providing Fermi's Chicago University lecture notes to me. I wish Fermi's Lecture Notes were available to the present generation of students.