Notes for Lectures on Quantum Mechanics *

Connection Between Coordinate and Momentum Representations

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We shall discuss the connection between the coordinate and momentum representations. The relation between the two representations is most conveniently displayed using the completeness formula

$$\int |x\rangle\langle x|\, dx = \hat{I}, \qquad \int |p\rangle\langle p|\, dp = \hat{I}. \tag{1}$$

So given a ket vector $|f\rangle$ how do you relate the representatives in the coordinate and momentum representations? This process is illustrated below. We begin with the wave function:

$$\begin{aligned} f(x) = \langle x|f\rangle &= \langle x|\hat{I}|f\rangle && (2)\\ &= \langle x| \int |p\rangle\langle p|dp\,|f\rangle && (3)\\ &= \int dp\,\langle x|p\rangle\langle p|f\rangle && (4)\\ &= \int dp\,\langle x|p\rangle\tilde{f}(p). && (5) \end{aligned}$$

This gives the desired relation between the coordinate space wave function $f(x)$ and the momentum space wave function $\tilde{f}(p)$. However, we still need to answer what are the transformation functions $\langle x|p\rangle$? This is easy to answer. For every vector $|f\rangle$, the inner product $\langle x|f\rangle$ represents that wave function of the state $|f\rangle$. So $\langle x|p\rangle$ is wave function of the state $|p\rangle$ which is just the eigenstate of momentum $\hat{p}$ with eigenvalue p. So this wave function $\langle x|p\rangle$ can be computed by solving the eigenvalue problem of the momentum operator in the coordinate representation, see next section.

*Updated:Sept26, 2021; Ver 1.x

Eigenfunctions of momentum operators

Delta function normalization Let $u_p(x)$ denote the wave function for the state with definite momentum p. Then

$$\hat{p}u_p(x) = pu_p(x), \Rightarrow -i\hbar\frac{du_p(x)}{dx} = pu_p(x). \tag{6}$$

The solution of this differential equation is easy to write and is given by

$$u_p(x) = C\exp(ipx/\hbar) \tag{7}$$

The normalization constant C is fixed by using the orthonormality property of the momentum eigenstates $\langle p''|p'\rangle = \delta(p''-p')$. Inserting a completeness identity $\int |x\rangle\langle x| = \hat{I}$ we get

$$\begin{aligned}
\delta(p''-p') &= \langle p''|p'\rangle && (8)\\
&= \langle p''|\Big(\int dx\,|x\rangle\langle x|\Big)|p'\rangle && (9)\\
&= \int dx\,\langle p''|x\rangle\langle x|p'\rangle && (10)\\
&= \int dx\,u^*_{p''}(x)u_{p'}(x) && (11)
\end{aligned}$$

Thus we get the normalization condition

$$\boxed{\int dx\,u^*_{p''}(x)u_{p'}(x) = \delta(p''-p')}. \tag{12}$$

Using this condition we can now fix the constant C.

$$\begin{aligned}
\int dxC^*\exp(-ip''x/\hbar)C\exp(ipx/\hbar) &= \delta(p''-p') && (13)\\
\Rightarrow 2\pi\hbar|C|^2 &= 1. && (14)
\end{aligned}$$

where in writing the last step the identity

$$\int dx\exp(i(p'-p'')x/\hbar) = 2\pi\hbar\delta(p''-p') \tag{15}$$

has been used. Thus we see that $C = 1/\sqrt{2\pi\hbar}$ and the normalised eigenfunctions of momentum are given by

$$u_p(x) = \frac{1}{\sqrt{2\pi\hbar}}\exp(ipx/\hbar). \tag{16}$$

The relation between the coordinate and momentum wave space functions becomes

$$f(x) = \frac{1}{\sqrt{2\pi\hbar}} \int \exp(ipx/\hbar)\tilde{f}(p)\, dp. \tag{17}$$

The inverse relation is easily written down

$$\tilde{f}(p) = \frac{1}{\sqrt{2\pi\hbar}} \int \exp(-ipx/\hbar) f(x)\, dx. \tag{18}$$

Box normalization The operators $\hat{x}$ and $\hat{p}$ are unbounded operators and we must deal with infinite dimensional vector spaces. This requires a deeper level of mathematics than that has been introduced so far. The steps, that are used to derive answers are all formal and carry no rigour, and are justified by the fact that the final results are correct and physically sensible. A time honoured rigorous approach begins with with working out everything by restricting x to a 'box' of finite size L and taking L to infinity at the end. We shall here discuss the box normalisation for momentum wave functions and give a set of working rules sufficient for later purposes.

Therefore to begin, we start with the Hilbert space of all square integrable functions over the interval $(-L/2, L/2)$. Further it is required that the functions in this space satisfy the periodic boundary condition

$$f(x+L) = f(x). \tag{19}$$

This ensures that the operators $\hat{x}, \hat{p}$ defined by

$$\hat{x} f(x) = x f(x), \qquad \hat{p} f(x) = -i\hbar \frac{df(x)}{dx} \tag{20}$$

are hermitian. Also they satisfy the required CCR. Next we seek the eigenvalues and eigenfunctions of the momentum operator $\hat{p}$ and and solution of the eigenvalue equation

$$-i\hbar \frac{du(x)}{dx} = pu(x) \tag{21}$$

is already found to be

$$u_p(x) = N \exp(ipx/\hbar). \tag{22}$$

Imposing the periodic boundary condition $u(x+L) = u(x)$ we get

$$\exp(ipL/\hbar) = 1 \Rightarrow pL/\hbar = 2\pi n, n = 0, \pm 1, \pm 2 \ldots. \tag{23}$$

Thus we have the momentum eigenfunctions and eigenvalues given by

$$u_{p_n}(x) = N\exp(ip_n x/\hbar), p_n = \frac{2\pi n\hbar}{L}, n = 0, \pm 1, \pm 2 \ldots . \tag{24}$$

The normalization is chosen to be

$$\int_{-L/2}^{L/2} |u_{p_n}(x)|^2 dx = 1. \tag{25}$$

This gives $N = \frac{1}{\sqrt{L}}$ and the normalized eigenfunctions are

$$u_{p_n}(x) = \frac{1}{\sqrt{L}}\, e^{ip_n x/\hbar} \tag{26}$$

The eigenfunctions of momentum satisfy the completeness relation

$$\sum_n u^*_{p_n}(x) u_{p_m}(x) = \delta(x-y). \tag{27}$$

The change of representation formula, relating the coordinate and momentum space wave functions assumes the form

$$\tilde{\psi}(p_n) = \int_{-L/2}^{L/2} u^*_{p_n}(x)\psi(x)dx, \tag{28}$$

and the inverse relation is

$$\psi(x) = \sum_n u_{p_n}(x)\tilde{\psi}(p_n). \tag{29}$$

Eigenvectors of position

We now seek eigenvectors of position operator $\hat{x}$. It is not difficult to see that no eigenvector exists in the conventional sense because if $f(x)$ is and eigenvector of $\hat{x}$, then we must have

$$\hat{x}f(x) = x_0 f(x) \tag{30}$$

$$\text{or } (x - x_0)f(x) = 0. \tag{31}$$

Thus $f(x)$ must vanish for all $x \neq x_0$ and will have zero norm. The eigenvalue equation has a formal solution $f(x) = \delta(x - x_0)$ which can be regarded as a 'generalized eigenvector'. All these difficulties are ultimately related to the fact that $\hat{x}$ is an unbounded operator in the space of square integrable functions and has no eigenvectors in strict mathematical sense.

qm-lec-01003 1.x Created : Aug 31, 2021 Printed : September 30, 2021 KApoor

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