Notes for Lectures on Quantum Mechanics *

Time Variation of Average Values

A. K. Kapoor
http://0space.org/users/kapoor
akkapoor@cmi.ac.in; akkhcu@gmail.com

Abstract

Assuming time development of states to be given by

$$i\hbar\frac{d|\psi,t\rangle}{dt} = H|\psi t\rangle,$$

an equation for time variation of average value of a dynamical variable is derived. Classical correspondence is used to identify the generator of time evolution with Hamiltonian. A dynamical variable not depending explicitly on time is a constant of motion if it commutes with the Hamiltonian.

The time evolution of a quantum system is governed by the Schrodinger equation

$$i\hbar\frac{d}{dt}|\psi t\rangle = \hat{H}|\psi t\rangle. \tag{1}$$

We will obtain an equation for time development of averages of a dynamical variable $\hat{F}$. The result will turn out to have an obvious correspondence with the classical equation of motion for dynamical variable F. This then will suggest the identification of $\hat{H}$ as the operator representing the Hamiltonian of the system.

Let $F(q,p,t)$ be an dynamical variable of the system and let $\hat{F}$ denote the corresponding operator. We are interested in finding out how the average value

$$\langle\hat{F}\rangle \equiv \langle\psi t|\hat{F}|\psi t\rangle \tag{2}$$

changes with time. The time dependence of the average value comes from dependence of the three objects, the operator$\hat{F}$, the bra vector $\langle\psi t|$, and the ket vector $|\psi t\rangle$, present in Eq.(2). The equation conjugate to the Schrodinger equation

$$i\hbar\frac{d}{dt}\,|\psi t\rangle = \hat{H}|\psi t\rangle \tag{3}$$

is given by

$$-\,i\hbar\frac{d}{dt}\,\langle\psi t| = \langle\psi t|\hat{H}^\dagger \tag{4}$$

Since the operator $\hat{H}$ is hermitian, the above equation takes the form

$$-\,i\hbar\frac{d}{dt}\,\langle\psi t| = \langle\psi t|\hat{H} \tag{5}$$

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Therefore

$$\frac{d}{dt}\langle\hat{F}\rangle = \left(\frac{d}{dt}\langle\psi t|\right)\hat{F}|\psi t\rangle + \langle\psi t|\frac{d\hat{F}}{dt}|\psi t\rangle + \langle\psi t|\hat{F}\left(\frac{d}{dt}|\psi t\rangle\right) \tag{6}$$

Using Eq.(3) and Eq.(5) in Eq.(6) we get

$$\frac{d}{dt}\langle\hat{F}\rangle = -\frac{1}{i\hbar}\langle\psi t|\hat{H}\hat{F}|\psi t\rangle + \langle\psi t|\frac{d\hat{F}}{dt}|\psi t\rangle + \frac{1}{i\hbar}\langle\psi t|\hat{F}\hat{H}|\psi t\rangle. \tag{7}$$

The above equation is rearranged to give the final form

$$\frac{d}{dt}\,\langle\hat{F}\rangle = \langle\frac{\partial}{\partial t}\hat{F}\rangle + \frac{1}{i\hbar}\langle\,[\hat{F},\hat{H}]\,\rangle. \tag{8}$$

This result is known as Ehrenfest theorem. Comparing Eq.(8) with the equation of motion in classical mechanics for time evolution of dynamical variables

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \{F,H\}_{PB} \tag{9}$$

and remembering that the commutator divided by $i\hbar$ corresponds to the Poisson bracket in the limit $\hbar \to 0$, we see that $\hat{H}$ must be identified as the operator corresponding to the Hamiltonian H of the system.