Notes for Lectures on Quantum Mechanics *

Stationary States and Constants of Motion

A. K. Kapoor
http://0space.org/users/kapoor
akkapoor@cmi.ac.in; akkhcu@gmail.com

Stationary states

Let us consider time evolution of a system if it has a definite value of energy at an initial time t_0. The value of the energy then has to be one of the eigenvalues and the state vector will be the corresponding eigenvector. So $|\psi t_0\rangle = |E_m\rangle$, then at time t the system will be in the state given by

$$|\psi t\rangle = U(t,t_0)|E_m\rangle = \exp(-iE_m(t-t_0)/\hbar)|E_m\rangle. \tag{1}$$

It must be noted that the state vector at different times is equal to the initial state vector times a *numerical phase factor* $(\exp(-iE_m(t-t_0)/\hbar))$. Therefore, the vector at time t represents the same state at all times. Such states are called **stationary states** because the state does not change with time. Every eigenvector of energy is a possible stationary state of a system. In such a state the average value of a dynamical variable, $\hat{X}$, not having time explicitly, is independent of time even if $\hat{X}$ does not commute with Hamiltonian. In fact the probabilities of finding a value on a measurement of the dynamical variable are independent of time.
Unless mentioned otherwise, we shall always assume that the Hamiltonian H of the system under discussion is independent of time.

Constant of motion

The time evolution of average value of an operator $\hat{F}$ is given by

$$\frac{d}{dt}\langle\psi t|\hat{F}|\psi t\rangle = \langle\psi t|\Big(\frac{\partial\hat{F}}{\partial t}\Big)|\psi t\rangle + \frac{1}{i\hbar}\langle\psi t|[\hat{F},\hat{H}]|\psi t\rangle \tag{2}$$

This equation corresponds to the classical Poisson bracket equation of motion for (classical) dynamical variable F

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \{F,H\}_{\text{PB}} \tag{3}$$

If a dynamical variable F does not contain explicit time dependence, then we have $\frac{\partial F}{\partial t} = 0$. If such an operator $\hat{F}$ commutes with the Hamiltonian operator $\hat{H}$, we will have

$$\boxed{[\hat{F},\hat{H}] = 0} \tag{4}$$

*qm-lec-09004-Updated:Sept 21, 2021; Ver 2.x and Eq.(2) shows that

$$\frac{d}{dt} \langle \psi t | \hat{F} | \psi t \rangle = 0$$

Therefore in an arbitrary state, the average value of $\hat{F}$ does not change with time. Such a dynamical variable will be called a **constant of motion**.