Notes for Lectures on Quantum Mechanics *

Solution of Time Dependent Schrödinger Equation

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A scheme to solve the time dependent Schrödinger equation

$$i\hbar\frac{d}{dt}|\psi\rangle = \hat{H}|\psi\rangle \tag{1}$$

is described. The final solution will be presented in the form, see Eq.(17)

$$|\psi t\rangle = U(t, t_0)|\psi t_0\rangle \tag{2}$$

For our present discussion, it will be assumed that the Hamiltonian $\hat{H}$ does not depend on time explicitly. Let the state vector of system at initial time $t = 0$ be denoted by $|\psi_0\rangle$.

Since $\hat{H}$ is always assumed to be hermitian, its eigenvectors form an orthonormal complete set and we can expand the state vector at time t, $|\psi t\rangle$, in terms of the eigenvectors. Denoting the normalized eigenvectors by $|E_n\rangle$, we write

$$|\psi t\rangle = \sum_n c_n(t)|E_n\rangle. \tag{3}$$

where the constants $c_n(t)$ are to be determined. Substituting (3) in (1), we get

$$i\hbar\frac{d}{dt}\sum_n c_n(t)|E_n\rangle \quad = \quad \hat{H}|\psi t\rangle \tag{4}$$

$$i\sum_n \hbar\frac{dc_n(t)}{dt}|E_n\rangle \quad = \quad \sum_n c_n(t)\hat{H}|E_n\rangle \tag{5}$$

Taking scalar product with $|E_m\rangle$ and using orthonormal property of the eigenvectors $|E_n\rangle$, we get

$$i\hbar\frac{dc_m(t)}{dt} = E_m c_m(t). \tag{6}$$

which is easily solved to give

$$c_m(t) = c_m(0)e^{-iE_m t/\hbar}. \tag{7}$$

Therefore, $|\psi t\rangle$,the solution of time dependent equation becomes

$$|\psi t\rangle = \sum_m c_m(0)e^{-iE_m t/\hbar}.|E_m\rangle. \tag{8}$$

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The coefficients $c_m(0)$ are determined in terms of the state vector $|\psi_0\rangle$ at time $t=0$ by setting time $t=0$ in the above equation. This gives

$$|\psi_0\rangle = \sum_n c_n(0)|E_n\rangle. \tag{9}$$

The unknown coefficients $c_n(0)$ can now be computed; taking scalar product of Eq.(9), with $|E_m\rangle$ we get

$$c_m(0) = \langle E_m|\psi_0\rangle. \tag{10}$$

Thus Eq.(8) and (10) give the solution of the time dependent Schrödinger equation as

$$\boxed{|\psi t\rangle = \sum_n c_n(0)\exp(-iE_n t/\hbar)|E_n\rangle}. \tag{11}$$

The right hand side of the above equation can be rewritten as

$$\begin{aligned}\sum_n c_n(0)\exp(-iE_n t/\hbar)|E_n\rangle &= \sum_n c_n(0)\exp(-iHt/\hbar)|E_n\rangle & (12)\\ &= \exp(-iHt/\hbar).\sum_n c_n(0)|E_n\rangle & (13)\end{aligned}$$

Therefore Eq.(11) takes the form

$$|\psi t\rangle = \exp(-iHt/\hbar)|\psi_0\rangle. \tag{14}$$

In general, if the state vector is know at time $t=t_0$, instead of time $t=0$, the result Eq.(14) takes the form

$$\begin{aligned}|\psi t\rangle &= \exp(-iH(t-t_0)/\hbar)\sum_n c_n(t_0)|E_n\rangle & (15)\\ &= \exp(-iH(t-t_0)/\hbar)|\psi t_0\rangle. & (16)\end{aligned}$$

The time evolution operator $U(t,t_0)$, of Eq.(2), is therefore given by

$$\boxed{U(t,t_0) = \exp(-iH(t-t_0)/\hbar)}. \tag{17}$$