Notes for Lectures on Quantum Mechanics
Schrödinger Picture —- Important Points

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Abstract

The time evolution of a general quantum system is reviewed in an abstract setting. The eigenstates of energy are seen to have all properties that make them qualify for being called stationary states.

The state vector at a given time specifies the state of the system at a given time and the state at any time is obtained by solving the Schrödinger equation.

$$i\hbar\frac{d|\psi t\rangle}{dt} = H|\psi t\rangle. \tag{1}$$

where H is the Hamiltonian operator. The reason for identification of H, in the above equation, with Hamiltonian is best brought out in by means of correspondence with equations in classical mechanics.

From now on we will assume that the Hamiltonian H does not depend on time. In this case the state vector at time t is related to the state vector at initial time t_0 by

$$|\psi t\rangle = U(t,t_0)|\psi t_0\rangle \tag{2}$$

where

$$U(t,t_0) = \exp\Big(-\frac{iH(t-t_0)}{\hbar}\Big) \tag{3}$$

Since H is a hermitian operator, it follows that $U(t,t_0)$ is a unitary operator.

What happens to (3) if H depends on time? GBox01

The Hamiltonian operator being Hermitian leads to the following important consequences. In the table below a few examples of time evolution of states are given.

Table : Time evolution energy eigenstates of a quantum system

S.N.	State at initial time	State at time t
1.	$\vert E_n\rangle$	$e^{-iE_nt/\hbar}\vert E_n\rangle$
2.	$c_1\vert E_n\rangle + c_2\vert E_n\rangle$	$c_1e^{-iE_nt/\hbar}\vert E_1\rangle + c_2e^{-iE_2t/\hbar}\vert E_n\rangle$
3.	$\sum_k c_k\vert E_k\rangle$	$\sum_k c_k e^{-iE_kt/\hbar}\vert E_k\rangle$
4.	states $\vert\psi t_0\rangle, \vert\phi t_0\rangle$ then $c_1\vert\psi t_0\rangle + c_2\vert\phi t_0\rangle$	evolve into $\vert\psi t\rangle, \vert\phi t\rangle$, evolves into $c_1\vert\psi t\rangle + c_2\vert\phi t\rangle$

- The first row in the table shows that the energy eigenstates

$$H|E_n\rangle = E_n|E_n\rangle \tag{4}$$

  i.e. the states corresponding to a definite value of energy, have a very simple time evolution. The state vector changes by phase factor, a multiplicative constant of absolute value 1. Thus the state itself does not change with time. Therefore, energy states are called *stationary states.*

- The time evolution preserves the superposition of states as is brought out by the examples in the second and last rows of the table.

- The time evolution is unitary and hence norm of the state vector is preserved. Mathematically this means that the norm $\langle\psi t|\psi t\rangle$ is independent of time. In other words

$$\langle\psi t|\phi t\rangle = \langle\psi t_0|\phi t_0\rangle \tag{5}$$

$$\text{and} \quad \frac{d\|\psi(t)\|}{dt} = 0 \tag{6}$$

  Remembering that $\|\psi(t)\|^2$ is just the sum of probabilities of all possible outcomes, The above result has a physical interpretation total probability of all possible outcomes of a measurement remains constant $(= 1)$ at all times.

Here the results given above are a consequence of Hamiltonian being hermitian.

Time development of average values

The average value of a dynamical variable $\hat{F}$ in a state $|\psi\rangle$ evolves according to the equation

$$\frac{d}{dt}\langle\psi t|\hat{F}|\psi t\rangle = \langle\psi t|\Big(\frac{\partial\hat{F}}{\partial t}\Big)|\psi t\rangle + \frac{1}{i\hbar}\langle\psi t|[\hat{F},\hat{H}]|\psi t\rangle \tag{7}$$

Remarks:
Following Dirac, in an alternate approach to time evolution, one can start from requirements that superposition be preserved and the normalization of the state vector should not change with time and prove that this leads to an equation of the form (1) where H some hermitian operator. Identification with operator corresponding to Hamiltonian can then be done by making use of classical correspondence.