Classical Mechanics Module Summary

The Action Principle

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Contents

§1 Lesson Objectives

1. To introduce configuration space of a system;
2. To introduce paths in configuration space and to define action as a functional of paths in configuration space;
3. To formulate Hamilton's action principle and and to prove that it is equivalent to Euler Lagrange equations of motion.
4. To explain Weiss action principle.

§2 Points to Recall and Discuss

The state of a system consisting of n particles and having r holonomic constraints can specified by a set of $N = n - r$ independent generalized coordinates $\mathbf{q} = (q_1, q_2, q_N)$.

The dynamics of the system is governed by Lagrangian L which is a `function` of $2N$ variables $\mathbf{q}, \dot{\mathbf{q}}$.

The Euler Lagrange equations of motion

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}_k}\Big) - \frac{\partial L}{\partial q_k} = 0. \tag{1}$$

describe the dynamics of a mechanical system.

§3 Configuration Space Action functional

§3.1 Configuration space

Let us consider a system with N degrees of freedom. At a given time t the system is completely specified by giving values the values of N generalised coordinates $q_1(t), \ldots, q_N(t)$ and their time derivatives.

We may arrange $q's$ in a row to form an N component vector

$$\mathbf{q} = (q_1, q_2, \ldots, q_N) \tag{2}$$

the N component vector can be represented by points in an N dimensional space called configuration space. Conversely a point in *configuration space* represents a possible set of values of $(q_1, \ldots, q_N)$

§3.2 Paths in configuration space

With time $\mathbf{q}$ change and so also does the position of the point representing the system. Thus with time, the point representing the system will trace out a path in configuration space. This path in configuration space is obtained by solving the Euler Lagrange equations of motion.

As Euler Lagrange equations are second order differential equations, the motion of the system, *i.e.* the state at any time, is completely known if we specify the initial values of $\mathbf{q}$ and $\dot{\mathbf{q}}$ at some time t_0.. With this as input, solving the Euler Lagrange equations of motion gives the generalised coordinates $\mathbf{q}(t)$ at all times, hence the path followed by system, in the configuration space is known.

§3.3 Specifying coordinates at the end points

To solve the Euler Lagrange equations for time t in the interval (t_1, t_2), one may give boundary conditions on $\mathbf{q}$ instead of initial conditions on $\mathbf{q}, \dot{\mathbf{q}}$ at time t_1. This means that coordinates $\mathbf{q}$ at the initial time t_1 and the final t_2 are to be specified. Thus we look for solution of Euler Lagrange equations.

$$\mathbf{q}(t) = \{q_1(t), \ldots, q_N(t)\} \quad \text{for} \quad t_1 \le t \le t_2 \tag{3}$$

subjected to conditions

$$\mathbf{q}(t_1) = \{q_1(t_1), \ldots, q_N(t_1)\}, \qquad \mathbf{q}(t_2) = \{q_1(t_2), \ldots, q_N(t_2)\}. \tag{4}$$

This amounts to asking what path is followed in configuration space, if we know the end points P_1 and P_2. Several paths in configuration space with fixed end points are shown below.

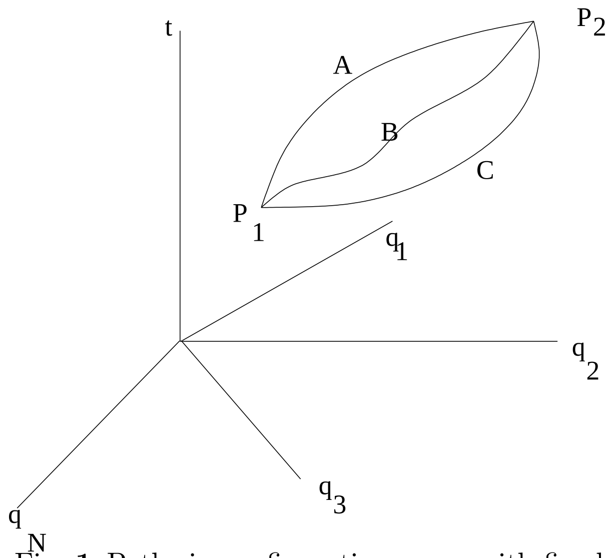

Fig. **1** Paths in configuration space with fixed end points

§3.4 The Actional Functional

Which path forms the solution of equations of motion? The answer given by the Hamilton's principle also known as the "Action Principle", is stated below. We first define action functional $\Phi[C]$ as a *functional of paths*. Given a path C, we know the coordinates as function of time. Along a given path the generalised velocities, $\dot{\mathbf{q}}$, at time t, are computed by taking time derivatives of the coordinates.

$$\dot{\mathbf{q}}(t) = \left\{ \frac{dq_1(t)}{dt}, \frac{dq_2(t)}{dt}, \ldots, \frac{dq_N(t)}{dt} \right\} \tag{5}$$

Thus the Lagrangian $L(\mathbf{q}, \dot{\mathbf{q}}, t)$, for a given path, is expressible as a function of time t. This function of time when integrated over t, from t_1 to t_2, defines the action functional [1] $\Phi[C]$ for the path C:

$$\Phi[C] = \int_{t_1}^{t_2} L(\mathbf{q}, \dot{\mathbf{q}}, t) dt. \tag{6}$$

Note that for a given system, the right hand is a number which depends on the path C, being different for different paths.

[1] A functional is a number assigned to function taken from class of functions. Here the functions are coordinates $q_k(t)$ as function of time.

Important:

✍ It is important to remember that the Lagrangian is a *function* of $2N$ independent variables $(\mathbf{q}, \dot{\mathbf{q}})$. Therefore for purpose of computing partial derivative of the Lagrangian w.r.t. q_k means that all $q_j, j \neq k$, and all $\dot{q}_m$ are held constants, with a similar statement for derivatives of L w.r.t. the generalized velocities. Thus

$$\frac{\partial L}{\partial q_k} \equiv \frac{\partial L}{\partial q_k}\Big|_{q_j,\dot{q}_m}, \qquad \frac{\partial L}{\partial \dot{q}_k} \equiv \frac{\partial L}{\partial \dot{q}_k}\Big|_{q_m,\dot{q}_j}, \text{where } j \neq k, \text{and all } m. \tag{7}$$

✍ The action functional is a *functional of paths* in the configuration space. Along a path in configuration space, the generalized coordinates $\mathbf{q}$ are some functions of time t. The generalized velocities along the path $\dot{\mathbf{q}}$ are to be computed by differentiating $\mathbf{q}(t)$ w.r.t. t.

$$\dot{q}_k(t) \equiv \frac{dq_k(t)}{dt}. \tag{8}$$

This may be rephrased as "for purpose of computing the action functional along a given path in configuration space, the generalized coordinates and generalized velocities are no longer independent."

§4 Hamilton's Principle

As already mentioned, a solution to the Euler Lagrange equations of motion requires specification of generalized coordinates and generalized velocities at initial time.

Alternatively, a solution to the Euler Lagrange equations can be obtained by specifying the generalized coordinates at the initial and final times t_1 and t_2. As we shall see below, the action principle formulates the path followed, between two end points, as being the path which makes the action extremum.

§4.1 Infinitesimally close paths

Let C be a given path connecting the points $\mathbf{q}_1$ at t_1 to point $\mathbf{q}_2$ times t_2. Let C' be another path which differs infinitesimally from the path C. The path C' starts from $\mathbf{q}_1'$ at time t_1' and ends at $\mathbf{q}_2'$ at time t_2'. Let the values of coordinates be $\mathbf{q}(t)$ and $\mathbf{q}'(t)$ for the two paths. We will say that C' is infinitesimally different from the path C, if the quantities defined by

$$\Delta t_1 = t_1' - t_1, \qquad \Delta t_2 = t_2' - t_2, \tag{9}$$

$$\Delta \mathbf{q}_1 = \mathbf{q}_1' - \mathbf{q}_1, \qquad \Delta \mathbf{q}_2 = \mathbf{q}_2' - \mathbf{q}_2, \tag{10}$$

and

$$\delta\mathbf{q}(t) = \mathbf{q}'(t) - \mathbf{q}(t), \qquad t_1 \leq t \leq t_2, \tag{11}$$

are infinitesimal quantities. For our present purpose, it will be unimportant weather we take (t_1, t_2), or (t_1', t_2'), as the range of t in equation Eq.(6).

The difference in velocities for the two paths is computed by using

$$\delta\dot{\mathbf{q}}(\mathbf{t}) \;=\; \frac{d}{dt}\mathbf{q}'(t) - \frac{d}{dt}\mathbf{q}(t) = \frac{d}{dt}(\delta\mathbf{q}(t)). \tag{12}$$

§4.2 Computing variation of action

To formulate Hamilton's principle, we compute variation of action functional when path is varied from C to C'

$$\Phi[C'] - \Phi[C] \tag{13}$$

$$= \int_{t_1'}^{t_2'} L(\mathbf{q}'(t), \dot{\mathbf{q}}'(t), t)dt - \int_{t_1}^{t_2} L(\mathbf{q}(t), \dot{\mathbf{q}}(t), t)dt \tag{14}$$

$$= \int_{t_1'}^{t_1} L(\mathbf{q}'(t), \dot{\mathbf{q}}'(t), t)dt + \int_{t_1}^{t_2} L(\mathbf{q}'(t), \dot{\mathbf{q}}'(t), t)dt + \int_{t_2}^{t_2'} L(\mathbf{q}'(t), \dot{\mathbf{q}}(t), t)dt - \int_{t_1}^{t_2} L(\mathbf{q}(t), \dot{\mathbf{q}}(t), t)dt \tag{15}$$

$$\approx (t_1 - t_1')L(\mathbf{q}(t_1), \dot{\mathbf{q}}(t_1), t) + \int_{t_1}^{t_2} \{L(\mathbf{q}'(t), \dot{\mathbf{q}}'(t), t) - L(\mathbf{q}(t), \dot{\mathbf{q}}(t), t)\}dt + (t_2' - t_2)L(\mathbf{q}_2, \dot{\mathbf{q}}_2, t_2) \tag{16}$$

What lies behind the last step? (17)

$$\approx -\Delta t_1 L(\mathbf{q}(t_1), \dot{\mathbf{q}}(t_1), t_1) + \Delta t_2 L(\mathbf{q}(t_2), \dot{\mathbf{q}}(t_2), t_2) + \int_{t_1}^{t_2} \{L(\mathbf{q}'(t), \dot{\mathbf{q}}'(t), t) - L(\mathbf{q}(t), \dot{\mathbf{q}}(t), t)\}dt \tag{18}$$

We substitute for $\mathbf{q}'(t)$ in the last term of (18)

$$\mathbf{q}'(t) = \mathbf{q}(t) + \delta\mathbf{q}(t), \tag{19}$$

and use the fact that the paths C' and C differ by infinitesimal amount to get

$$\int_{t_1}^{t_2} [L(\mathbf{q}', \dot{\mathbf{q}}', t) - L(\mathbf{q}, \dot{\mathbf{q}}, t)]\ dt \tag{20}$$

$$= \int_{t_1}^{t_2} \left[L\Big(\mathbf{q} + \delta\mathbf{q}, \dot{\mathbf{q}}(t) + \frac{d}{dt}\delta\mathbf{q}(t)\Big) - L(\mathbf{q}, \dot{\mathbf{q}}, t)\right]\ dt \tag{21}$$

$$\approx \int_{t_1}^{t_2} \sum_k \Big(\frac{\partial L}{\partial q_k}\delta q_k + \frac{\partial L}{\partial \dot{q}_k}\delta\dot{q}_k\Big)dt + \text{second order terms} \tag{22}$$

$$= \int_{t_1}^{t_2} \sum_k \left(\frac{\partial L}{\partial q_k}\delta q_k - \frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}_k}\Big)\delta q_k\right)\ dt + \sum_k \frac{\partial L}{\partial \dot{q}_k}\delta q_k\Big|_{t_1}^{t_2} \tag{23}$$

Integration by parts has been done in the second term (24)

Substituting (23) in (18) we get

$$\Phi[C'] - \Phi[C] \approx \int_{t_1}^{t_2} \Big(\sum_k \frac{\partial L}{\partial q_k} - \frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}_k}\Big)\Big)\delta q_k + \Big[L\Delta t + \sum_k \frac{\partial L}{\partial \dot{q}_k}\delta q_k\Big]_{t_1}^{t_2} \tag{25}$$

✍ It must be remembered that so far we have kept the variation paths to be general one; there is no restriction that the paths must have the same end points.

§4.3 Condition for an extremum

We first consider special class of variations of path which keep the end points fixed, *i.e.*

$$\Delta t_1 = \Delta t_2 = 0 \tag{26}$$

$$\Delta q_k(t_1) = 0; \qquad \Delta q_k(t_2) = 0. \tag{27}$$

For such variations we get

$$\Delta\Phi(C) = \int_{t_1}^{t_2} \sum_k \Big(\frac{\partial L}{\partial q_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k}\Big)\delta q_k(t)dt. \tag{28}$$

Since the generalized coordinates are independent and the variations $\delta\mathbf{q}$ are arbitrary, right the right hand side of (28) vanishes if and only if the Euler Lagrange equations, ,

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}_k}\Big) - \frac{\partial L}{\partial q_k} = 0, \qquad \text{Write all steps from the last equation.}, \tag{29}$$

are satisfied *i.e.* $\mathbf{q}(t)$ is solution of EOM. This is summarised into the following statement of action principle.

§4.4 Action Principle — The Statement

Given configurations $\mathbf{q}_1, \mathbf{q}_2$ at times t_1 and t_2, the actual dynamical path C followed by a system is that for which the action is stationary *i.e.* C is that path about which infinitesimal variations, *with fixed end points*, do not produce any change in Φ. If C' is any other path infinitesimally close to C, then

$$\delta\Phi = \Phi[C'] - \Phi[C] = 0.$$

In other words, the action is stationary for the actual trajectory followed by the the system.

§5 Weiss Action Principle

This principle is about characterizing the paths in terms of general variations of the action, when the end points may not be fixed. It states that the dynamical path C followed by the system is such that the variations about the path produce only end point contribution to the variation in action

$$\begin{aligned}\Delta\Phi[C] &= \Delta \int_{t_1}^{t_2} L(q, \dot{q}, t)dt \\ &= \sum_k \Big(\frac{\partial L}{\partial \dot{q}_k}\Big)\delta q_k - H\Delta t\Big|_{t_1}^{t_2}.\end{aligned}$$

where

$$H = \sum_k \Big(\frac{\partial L}{\partial \dot{q}_k}\Big)\dot{q}_k - L \tag{30}$$

is called the Hamiltonian of the system. We skip the details of the steps leading to the Weiss action principle.

To check steps for Weiss action principle

§6 EndNotes

1. For action principle see [1, 4, 2, 3]; Weiss action principle can be found in [3].
2. For an account of different variational principles and historical developments see [5, 6]

§7 Anti Black Boxes

Eq.(17)**:** A result that

$$\int f(x)dx \approx (b-a)f(x), \qquad a < x < b$$

has been used. Try to formulate a precise statement of this result.

Closest to this 'vague' statement is the mean value theorem for integrals.

If f is a continuous function on the closed, bounded interval $[a, b]$, then there is at least one number c in (a, b) for which

$$f(c) = 1/(b-a)Int(f(t), t = a..b).$$

Ref1: Mean Value Theorem ;

Ref2: First mean value theorem for definite integrals.

Eq.(29)**:**]We give some details here. Here you have an equation of the form

$$\int_{t_1}^{t_2} F_k(t)\delta q_k(t)dt = 0. \tag{31}$$

where $\delta q_k(t)$ are arbitrary and independent functions. You are required to prove $F_k(t) = 0$ for all $t_1 < t < t_2$.

(a) Since $\delta q_k(t)$ are arbitrary and independent functions, you are allowed to choose $\delta q_k(t) = F_k(t)$. This gives

$$\int_{t_1}^{t_2} \sum_k |F_k(t)|^2 dt = 0. \tag{32}$$

Assuming $F_k(t)$ to be continuous functions of t, (32) holds if and only if

$$\sum_k |F_k(t)|^2 = 0$$

A sum of positive terms can be zero if and only if each term vanishes. This gives us Eq.(29).

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