# Lessons in Classical Mechanics

## Conservation of Energy

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## Contents



## 1 Objectives

To show that energy conservation law holds for systems if the Lagrangian is independent of time.

## 2 Let's recall and discuss

1. Give an example of application of energy conservation.

2. What do you understand by conservation law? When we say something $X(q,p)$ is a constant of motion what exactly it means mathematically?

3. Give examples of a system for which energy is not conserved.

4. Consider an example of a particle moving in one dimension in a potential $V(x)$. Taking total time derivative of total energy, $E = \frac{1}{2}m\dot{x}^2 + V(x)$ we would get

$$\frac{dE}{dt} = m\dot{x}\ddot{x} + \frac{dV(x)}{dx}\frac{dx}{dt}. \tag{1}$$

   The right hand side does not seem to become zero!
   What is missing?
   How do we see that the right hand side of (1) is zero?

## 3 Energy conservation

If the Lagrangian does not depend on time explicitly, there is a conservation law and the corresponding conserved quantity will be called as Hamiltonian. The Hamiltonian coincides with energy ($= KE + PE$) for a mechanical systems. For other physical systems also it qualifies to be identified with energy.

If Lagrangian does not contain $t$ explicitly

$$\frac{\partial L}{\partial t} = 0 \tag{2}$$

$$and \qquad \frac{dL}{dt} = \sum_k \frac{\partial L}{\partial q_k}\dot{q}_k + \sum_k \frac{\partial L}{\partial \dot{q}_k}(\ddot{q}_k) \tag{3}$$

using Euler Lagrange EOM we get

$$\frac{dL}{dt} = \sum_k \Big[\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_k})\dot{q}_k + \frac{\partial L}{\partial \dot{q}_k}\ddot{q}_k\Big] \tag{4}$$

$$= \frac{d}{dt}\sum_{k=1} \frac{\partial L}{\partial \dot{q}_k}\dot{q}_k \tag{5}$$

or

$$\frac{d}{dt}\Big(\sum_k \frac{\partial L}{\partial \dot{q}_k}\dot{q}_k - L\Big) = 0 \tag{6}$$

Hence $H$ defined by

$$H \stackrel{\text{def}}{\equiv} \sum_{k=1} \frac{\partial L}{\partial \dot{q}_k}\dot{q}_k - L \tag{7}$$

is a constant of motion we can also write

$$H = \sum_{k=1}^{N} p_k\dot{q}_k - L \tag{8}$$

Where $p_k = \frac{\partial L}{\partial \dot{q}_k}$ is called the **canonical momentum conjugate** to the coordinate $\dot{q}_k$and $H$ will be called **Hamiltonian** of the system

In an alternate form of dynamics, the canonical momenta take over the role played by velocities and Hamiltonian becomes central quantity which governs the dynamics.The EOM can be written in an alternate form called the Hamiltonian EOM. In the Hamiltonian dynamics the velocities are eliminated in favour of canonical momenta.

**An Example:** Let us consider a single particle moving in force field described by potential energy $V(\vec{r})$. Then

$$L = \frac{1}{2}m\dot{\vec{r}}^2 - V(\vec{r}); \qquad \vec{r} = (x, y, z) \tag{9}$$

the canonical momenta are

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}; \qquad p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y}; \qquad p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z}, \tag{10}$$

and the Hamiltonian is given by

$$H = \sum p_k\dot{q}_k - L \tag{11}$$

$$= p_x m\dot{x} + p_y m\dot{y} + p_z m\dot{z} - L \tag{12}$$

$$= m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - [\frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(\vec{r})] \tag{13}$$

$$= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + V(\vec{r}) \tag{14}$$

$$= \frac{1}{2}m\dot{\vec{r}}^2 + V(\vec{r}) \tag{15}$$

Thus the canonical momenta, in this example, coincide with the components of momentum $m\dot{\vec{r}}$ and Hamiltonian is equal to the energy. However, it must be remarked that *the canonical momenta are not always equal to 'ordinary' momenta and Hamiltonian need not be a sum of K.E + P.E.*

This is the case when the system is described, for example, by a velocity dependent generalized potential. Motion of a charged particle in external magnetic field constitutes an example of this type where the canonical momentum is not equal to ordinary momentum.

## 4 EndNotes

### Remarks

- ✍ The canonical momentum is in general not the same as momentum of a particle.
- ✍ The Hamiltonian is defined as

$$H = \sum_k p_k \dot{q}_k - L(q, \dot{q}, t) \tag{16}$$

  where $p_k$ is *canonical momentum* conjugate to $q_k$.
- ✍ For conservative mechanical systems the Hamiltonian coincides the energy of the system.
- ✍ The Hamiltonian is just the total energy of a system. It is conserved when the Lagrangian does not have explicit time dependence.

Every book on classical mechanics will have a discussion of energy conservation, see for Landau Lifshitz [1] and Calkin [2].

# References


[1] Landau, L. D. and Lifshitz E. M.,*Mechanics*, Volume 1 of Course of Theoretical Physics, Butterworth-Heinenann Linacre House, Jordan Hill, Oxford 3rd Ed.(1976).
We have closely followed the treatment by Landau Lifshitz in §6.

[2] Calkin, M.G. *Lagrangian and Hamiltonian Mechanics*, World Scientific Publishing Co. Pte. Ltd. (1996);
Here the energy conservation is discussed as an application of Noether's theorem to time translations.