Lessons in Classical Mechanics *

General Properties of Motion in Spherically Symmetric Potentials

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Contents

§1 Lesson objectives

The qualitative features of motion in a spherically symmetric potential are discussed with focus on

1. Nature of orbits; Accessible region; planar orbits for a spherically symmetric potential. Bounded and unbounded motion.

2. Condition(s) for circular orbits; Stability of circular orbits.

3. Fall to centre.

4. Escape to infinity.

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§2 Let's recall and discuss

We first briefly recall method used for qualitative discussion of motion in one dimension. Many of these ideas will be used here for radial motion in three dimensions.

Simplest physical systems of interest, for example Sun and Earth system, are two body systems interacting via a potential dependent only the distance between the two bodies only. This problem can be reduced to an equivalent one body problem in a spherically symmetric potential.

§2.1 Recall and Discuss — One dimension

Equilibrium points

If a particle, moving in one dimension, is released from rest at some point, it will in general move towards lower potential energy. If it is released at a minimum or maximum of the potential, it will remain at rest It therefore follows that at these points $\dot{x} = \ddot{x} = 0$.

Let x_0 be a point where the particle in equilibrium.

If the point x_0 is a minimum of the potential and the particle is disturbed slightly, it will execute oscillations about the minimum. In this case we say that the point x_0 is a point of {it stable equilibrium}.

If the equilibrium point is a maximum of the potential and even a slightest disturbance will make the particle move away from the equilibrium. In this case we say that the *equilibrium is unstable.*

Turning points

1. A particle moving in a potential cannot go to regions where its energy is less than the potential energy. Its motion is confined to those values of x where

   $$V(x) \leq E$$

   . To see this note that we must have

   $$\begin{aligned} E &= \frac{p^2}{2m} + V(x) && (1) \\ \therefore E &\geq V(x) && (2) \end{aligned}$$

   $$\because \boxed{\text{K.E.} = \tfrac{p^2}{2m} > 0}.$$

2. The region, the set of values of x where (1) holds, is called *classically accessible region.*

3. The points where $E = V(x)$ are called *turning points.* At a turning point the velocity becomes zero, $\dot{x} = 0$.

Range of energies for bounded motion

Assuming a continuous potential $V(x)$, the potential will have a minimum or a maximum between two turning points. For a given energy, a particle will execute a bounded motion if the potential has two turning points such that it has a minimum between the turning points.

In general, whether the motion is bounded, or unbounded, depends on the initial conditions and energy of the particle.

For all possible initial conditions the motion is bounded only if the energy is less that the values of potential $V(\pm\infty)$ and if both $V(\pm\infty)$ are greater than the absolute minimum of the potential.

§2.2 Recall and discuss — Reduction of two body problem

The Lagrangian of two body system

$$L = \frac{1}{2}m_1\Big(\frac{d\vec{x}_1}{dt}\Big)^2 + \frac{1}{2}m_2\Big(\frac{d\vec{x}_2}{dt}\Big)^2 - V(\vec{x}_1 - \vec{x}_2) \tag{3}$$

decomposes as sum of Lagrangian for the centre of mass and the for the relative motion:

$$\begin{aligned} L &= L_{\text{cm}} + L\text{rel}, && (4)\\ L_{\text{cm}} &= \frac{1}{2}M\Big(\frac{d\vec{X}_{\text{cm}}}{dt}\Big)^2, && (5)\\ L_{\text{rel}} &= \frac{1}{2}\mu\Big(\frac{d\vec{x}}{dt}\Big)^2 - V(\vec{r}). && (6)\end{aligned}$$

Here $M = m_1 + m_2$ is the total mass, and$\mu = \dfrac{m_1 m_2}{m_1 + m_2}$ is called reduced mass of the two body system. The energy of the relative motion is given by

$$E = \frac{1}{2}\mu\Big(\frac{d\vec{x}}{dt}\Big)^2 + V(\vec{r}). \tag{7}$$

§3 Effective potential for radial motion

§3.1 Spherically symmetric potential— Conservation laws

Consider motion of a particle is a spherically symmetric potential taking the Lagrangian of a system to be given by

$$\mathcal{L} = \frac{1}{2}\mu\dot{\vec{r}}^2 - V(r) \tag{8}$$

The Lagrangian does not contain time explicitly, hence we obtain energy conservation

$$\frac{1}{2}\mu\dot{\vec{r}}^2 + V(r) = E \qquad \text{(constant)}. \tag{9}$$

The Lagrangian is also invariant under rotations about any axis and in particular about the coordinate axes. This gives us conservation of angular momentum. Thus we have

$$\vec{L} = \mu\vec{r}\times\vec{v} = \text{constant of motion} \tag{10}$$

§3.2 Radial motion

We shall now make use of the conservation laws to give solution of motion in a spherically symmetric potential to quadratures.

Since $\vec{L} = \mu\vec{r} \times \vec{v}$ is a constant of motion the magnitude as the direction of $\vec{L}$ does not change with time. Also $\vec{r}$ and $\vec{v}$ always perpendicular to $\vec{L}$ which points in a fixed direction. Hence $\vec{r}$ and $\vec{v}$ remain in the plane perpendicular to $\vec{L}$. Therefore for a particle in a spherically symmetric potential, the motion is confined to a plane.

If $\vec{L}$ is zero, then $\vec{r} \times \vec{v} = 0$ and $\vec{r}$ will always be parallel to $\vec{v}$ and the particle moves in a straight line.

We, therefore, start with Lagrangian for a particle in two dimensions in plane polar coordinates

$$\mathcal{L} = \frac{1}{2}\mu\dot{\vec{r}}^2 + \frac{1}{2}\mu r^2\dot{\phi}^2 - V(r). \tag{11}$$

§3.3 Energy conservation

The expression for energy, associated with relative motion, given by

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\phi}^2 + V(r). \tag{12}$$

The energy is conserved because the Lagrangian (11) is independent of time.

§3.4 ϕ is a cyclic coordinate

Since $\dot{\phi}$ is a cyclic coordinate we have

$$\frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \mu r^2\dot{\phi} = \text{constant, say} L. \tag{13}$$

$\mu r^2\dot{\phi}$ is in fact seen to be equal to the magnitude of angular momentum.

§3.5 Effective potential

The velocity $\dot{\phi}$ can be eliminated using

$$\dot{\phi} = \frac{L}{2\mu r^2} \tag{14}$$

Making use of (12) and (14) we get,

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{L^2}{2\mu r^2} + V(r) = \frac{1}{2}\mu\dot{r}^2 + V_{\text{eff}}(r) \tag{15}$$

where we have introduced the notation

$$\boxed{V_{\text{eff}}(r) = V(r) + \frac{L^2}{2\mu r^2}}. \tag{16}$$

§4 General properties of radial motion

The radial motion in three dimensions in a spherically symmetric potential is just like motion in one dimension. The following discussion of motion in one dimension can be usefully extended to the radial motion in a spherically symmetric potential, if we replace the potential by the effective potential.

$$V(x) \longrightarrow V_{\text{eff}}(r) = V(r) + \frac{L^2}{2mr^2} \tag{17}$$

Thus for motion in three dimensional spherically symmetric potential

$$E = \frac{1}{2}m\dot{r}^2 + V_{\text{eff}}(r). \tag{18}$$

§4.1 Nature of Orbits

1. Conservation of angular momentum implies the motion of particle is in a plane

2. If angular momentum is zero, $\vec{r} \times \vec{p} = 0. \Longrightarrow \vec{r}$ and $\vec{p}$ are parallel. In this case the particle moves in a straight line.

3. If angular momentum is nonzero, the bounded or unbounded nature of the orbit can be decided in the same fashion as in one dimension by looking at the plot of the *effective potential.*

4. The motion is always confined to region where

$$E \geq V_{\text{eff}}(r). \tag{19}$$

   For a bounded motion, r varies between two extreme values r_1 and r_2, which correspond to the turning points. For these points the radial velocity becomes zero and the total energy is given by $V_{\text{eff}}(r_1) = V_{\text{eff}}(r_2) = E$.

§4.2 Circular orbits

1. The equilibrium in one dimension correspond to a fixed value of $x(t) =$ constant and $\dot{x} = \ddot{x} = 0$ for all times. In three dimensions $r(t) = \text{constant}, R$ corresponds to a circular orbit. For a circular orbit of radius R, we will have $\dot{r} = \ddot{r} = 0$ for all times. Thus the radius of a circular orbit is given by minima and maxima of the effective potential, (use (18)).

$$\left.\frac{dV_{\text{eff}}(r)}{dr}\right|_{r=R} = 0. \tag{20}$$

2. $r =$ constant. and angular momentum conservation $mr^2\dot{\phi} =$ constant imply

$$\dot{\phi} = \text{constant}.$$

   Thus the particle moving in a circular orbit has a constant constant angular velocity.

3. For a bounded motion we will have $r_1 < r < r_2$ and for a circular orbit of radius R, we must have $r_1 = r_2 = R$ and the energy is given by $E = V_{\text{eff}}(R)$. See Fig.1.

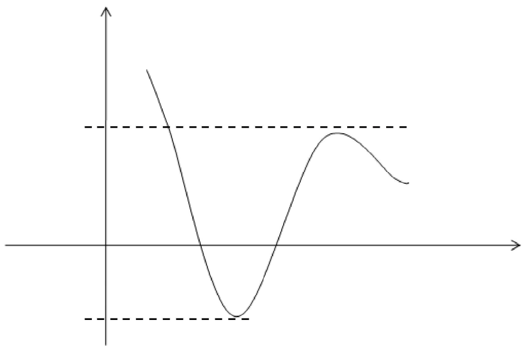

Fig. **1**

§4.3 Stability of circular orbits

For a circular orbit radial velocity and acceleration are zero. Hence $m\ddot{r} = 0 \Longrightarrow \frac{\partial}{\partial r}\left(V(r) + \frac{L^2}{mr^2}\right) = 0$. Thus the circular orbits correspond to the maxima and minima of the effective potential $V_{\text{eff}}(r)$.

The maximum corresponds to an unstable circular orbit orbit. The minimum corresponds to a stable circular orbit.

$$r = r_0 + \eta$$

§4.4 Fall to center

Let V(r) be finite as r $\longrightarrow$ 0. Then $V_{\text{eff}}(r) \longrightarrow \infty$ as $r \longrightarrow 0$ and a particle cannot reach r=0 for any value of E. However, for certain singular potentials the particle can reach center. Consider, for example, the case of a potential $V = \frac{-g}{r^4}, \quad g > 0$. Then the effective potential is

$$V_{\text{eff}} = \frac{-g}{r^4} + \frac{L^2}{2mr^2} \tag{21}$$

A sketch of the effective potential is shown in Fig.2. If $E >$ maximum of V_{eff}, then a particle coming from large distance can fall to center.

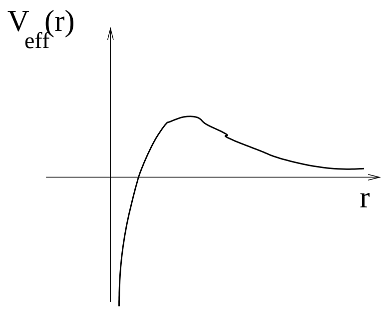

Fig. **2** Fall to centre

§4.5 Escape to ∞

Assume $V(r) \longrightarrow 0 \; as \; r \longrightarrow \infty$

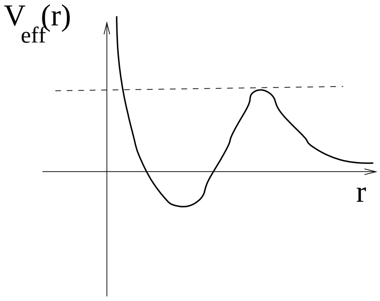

Fig. **3** Escape to infinity

A particle moving out can escape to infinity if $E >$ maximum of V_{eff} for $r > 0$.

§5 EndNotes