Lecture Notes in Classical Mechanics

Action Principle

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§1 Configuration space

Let us consider a system with N degrees of freedom. At a given time the system is completely specified by giving values the values of N generalised coordinates $q_1(t), \ldots, q_1(N)$. We may arrange $q's$ in a row to form an N component vector

$$q = (q_1(t), \ldots, q_N(t)) \tag{1}$$

the N component vector can be represented by points in an N dimensional space called configuration space. Conversely a point in *configuration space* represents a possible set of values of $(q_1, \ldots, q_N)$

With time q' change and also does the position of the point representing the system. Thus with time, the point representing the system will trace out a path in configuration space. As Euler Lagrange equations are second order differential equations, the motion of the system, the state at any time is completely known if we specify the initial values of q and $\dot{q}$ at some time t_0.. With this as input solving the Euler Lagrange equations of motion give the generalised coordinates $q(t)$ at all times hence the path followed by the system in the configuration space is known.

We say that possible states of system are given by points in configuration space and a set of generalised velocities. Knowing the state at initial time, the state at a later time is given by the solution of equations of motion.

§2 Paths in configuration space

An equivalent way of specifying the motion completely is to give the coordinates q at two different times t_1 and t_2. Thus we are looking for solution of Euler Lagrange equations.

$$q(t) = (q_1(t), \ldots, q_N(t)) \quad \text{for} \quad t_1 \leq t \leq t_2 \tag{2}$$

when their values at the initial time and final time

$$q(t_1) = (q_1(t_1), \ldots, q_N(t_1)), \qquad q(t_2) = (q_1(t_2), \ldots, q_N(t_2)) \tag{3}$$

are known. This amounts to asking what path is followed in configuration space, if we know the end points P_1 and P_2. Several paths in configuration space with fixed end points are shown below.

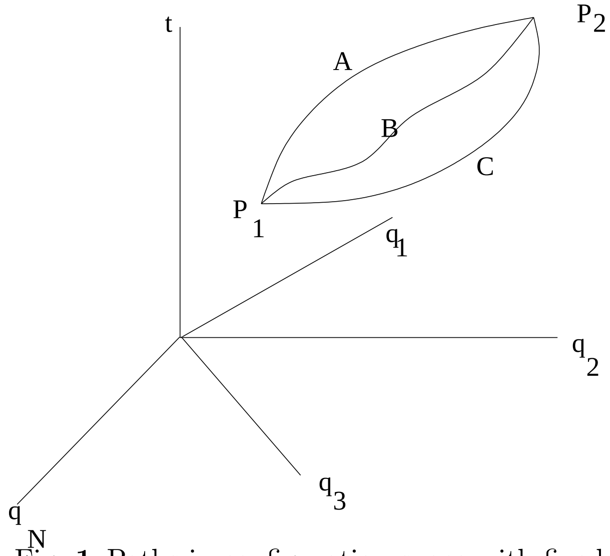

Fig. **1** Paths in configuration space with fixed end points

✍ It is important to remember that the Lagrangian is a *function* of two independent variables $q_k, \dot{q}_k$.

✍ The action functional is functional of paths in the configuration space. Along a path in configuration space q_k are some functions of time t and $\dot{q}_k$ is to be computed differentiating $q_k(t)$ w.r.t. t. This may be rephrased as "the generalized coordinates and generalized velocities are no longer independent."

§3 Action functional

The answer given by the Hamilton's principle also known as the "Action Principle", is stated below. We first define action functional $\Phi(C)$ or (S_C). Given a path C, we known the coordinates as function of time and also the generalised velocities.at times between t_1 and t_2. Thus the Lagrangian $L(q, \dot{q}, t)$, for a given path, is expressible as a function of time t. This function of time when integrated over from t_1 to t_2 defines the action functional $\Phi(c)$ for the path C: [1]

$$\Phi(C) = \int_{t_1}^{t_2} L(q, \dot{q}, t)dt. \tag{4}$$

Note that for a given system, the right hand is a number which depends on the path C, being different for different paths.

§4 Infinitesimally close paths

Now let C be a path which differs infinitesimally from the path C. The path C' starts from q' at time t_1' and ends at q_2' at time t_2'. Let the values of coordinate be

[1]A functional is a number assigned to function taken from class of functions. Here the functions are coordinates $q_k(t)$ as function of time.

$q(t)$ at times between t'_1 and t'_2. We will say that C' is infinitesimally different from the path C if the quantities defined by

$$\Delta t_1 = t'_1 - t_, \qquad \Delta t_2 = t'_2 - t_2, \tag{5}$$

$$\Delta q_1 = q'_1 - q_1, \qquad \Delta q_2 = q'_2 - q_2, \tag{6}$$

and

$$\delta q_1(t) = q'_1(t) - q(t), \qquad t_1 \le t \le t_2, \tag{7}$$

are infinitesimal quantities. For our present purpose, it is unimportant weather we take (t_1, t_2) or (t'_1, t'_2) as the range of t in equation Eq.(4). The difference in velocities for the two paths is computed by using

$$\begin{aligned} \delta\dot{q}(1) &= \frac{d}{dt}q'(t) - \frac{d}{dt}q(t) && (8)\\ &= \frac{d}{dt}(\delta q(t)). && (9) \end{aligned}$$

§5 Computing variation of action

To formulate Hamilton's principle, we compute variation of action functional when path is varied from C to C'

$$\begin{aligned} &\Phi(C') - \Phi(C) && (10)\\ &= \int_{t'_1}^{t'_2} L(q'_1(t), \dot{q}(t), t)dt - \int_{t_1}^{t_2} L(q(t), \dot{q}(t), t)dt && (11)\\ &= \int_{t'_1}^{t_1} L(q'(t), \dot{q}'(t), t)dt + \int_{t_1}^{t_2} L(q', \dot{q}', t)dt + \int_{t_2}^{t'_2} L(q', \dot{q}, t)dt \\ &\qquad - \int_{t_1}^{t_2} L(q, \dot{q}, t)dt && (12)\\ &\approx (t_1 - t'_1)L(q(t_1), \dot{q}(t_1), t) + \int_{t_1}^{t_2} \{L(q', \dot{q}', t) - L(q, \dot{q}, t)\}dt \\ &\qquad +(t_2 - t'_2)L(q_2, \dot{q}_2, t_2) && (13)\\ &\approx -\Delta t_1 L(q(t_1), \dot{q}(t_1), t_1) + \Delta t_2 L(q(t_2), \dot{q}(t_2), t_2 \\ &\qquad + \int_{t_1}^{t_2} \{L(q'(t), \dot{q}'(t), t) - L(q(t), \dot{q}(t), t)\}dt \\ & && (14) \end{aligned}$$

We substitute for $q'(t)$ in the last term of (14)

$$q'(t) = q(t) + \delta q(t), \tag{15}$$

and use the fact that the paths C' and C differ by infinitesimal amount to get

$$\begin{aligned}
&\int_{t_1}^{t_2} [L(q',\dot{q}',t) - L(q,\dot{q},t)]\ dt && (16)\\
&= \int_{t_1}^{t_2} \Big[L\Big(q+\delta q, \dot{q}(t) + \frac{d}{dt}\delta q(t)\Big) - L(q,\dot{q},t)\Big]\ dt && (17)\\
&\approx \int_{t_1}^{t_2} \sum_k \Big(\frac{\partial L}{\partial q_k}\delta q_k + \frac{\partial L}{\partial \dot{q}_k}\delta \dot{q}_k\Big) dt + \text{second order terms} && (18)\\
&= \int_{t_1}^{t_2} \sum_k \Big(\frac{\partial L}{\partial q_k}\delta q_k - \frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}_k}\Big)\delta q_k\Big)\ dt + \sum_k \frac{\partial L}{\partial \dot{q}_k}\delta q_k\Big|_{t_1}^{t_2} && (19)\\
&\boxed{\text{Integration by parts has been done in the second term}} && (20)
\end{aligned}$$

Substituting (19) in (14) we get

$$\Phi(C') - \Phi(C) \approx \int_{t_1}^{t_2} \Big(\sum_k \frac{\partial L}{\partial q_k} - \frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}_k}\Big)\Big)\delta q_k + \Big[L\Delta t + \sum_k \frac{\partial L}{\partial \dot{q}_k}\delta q_k\Big]_{t_1}^{t_2} \tag{21}$$

✍ It must be remembered that in the variations in the action along a path as computed in (21), there is no restriction on the infinitesimally close paths.

§6 Hamilton's Principle

We first consider special class of variations of path which keep the end points fixed, *i.e.*

$$\Delta t_1 = \Delta t_2 = 0 \tag{22}$$

$$\Delta q_k(t_1) = 0; \qquad \Delta q_k(t_2) = 0. \tag{23}$$

For such variations we get

$$\Delta\Phi(C) = \int_{t_1}^{t_2} \sum_k \Big(\frac{\partial L}{\partial q_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k}\Big)\delta q_k(t) dt. \tag{24}$$

It is now seen that the right the right hand side of (24) vanishes when ever Euler Lagrange equations are satisfied *i.e.* $q(t)$ is solution of EOM and the variation of action is zero implies that Euler Lagrange equations are satisfied. This is summarised into the following statement of action principle.

Action Principle Given the configurations q_1, q_2 at times t_1 and t_2, the actual dynamical path C followed by a system is that for which the action is stationary *i.e.* C is that path about which infinitesimal variations do not produce any change in Φ

$$\delta\Phi = \Phi(C') - \phi(C) = 0.$$

Note that the variation in path should not change the end points of the path.

§7 Weis Action Principle

This principle is about characterizing the paths in terms of general variations of the action, when the end points may not be fixed. It states that the dynamical path followed by the system is such that the variations about the classical trajectory have only end point contribution

$$\Delta\Phi(C) = \Delta \int_{t_1}^{t_2} L(q, \dot{q}, t)dt$$

$$= \sum_k \left(\frac{\partial L}{\partial \dot{q}_k}\right)\delta q_k - H\Delta t\Big|_{t_1}^{t_2}.$$