Mathematical Physics

The First Steps

Pankaj Sharan

and

Ashok K. Kapoor

Preface

The twentieth century saw great discoveries and growth in all sciences, particularly physics. The amount of core, or basic physics a student has to learn before reaching a research or advanced graduate level has become several times than what used to be about fifty years ago. Despite this, the number of years available to a student after 10+2 school level years has remained the same: three years at undergraduate level and two at the graduate or MSc level.

In order to cope with this situation, universities have been including more and more of the recent developments in the syllabi at the cost of the fundamental concepts and methods which require more time to absorb. The students quickly learn the tricks to somehow manage, hoping to get a fuller understanding later on. That hope, sadly, is not fulfilled due to demands of work.

It has been our experience that with the inclusion of more content, a great harm is done to the teaching of basic mathematical and experimental skills to the physics student. Of these two, learning mathematics demands more time.

Usually, there is just one course on mathematical methods in the initial semester of the 4-semester MSc Physics program. The topics included in the course are usually chosen for immediate use in the core physics part, while new tools are needed in teaching the advanced topics. This leads to a basic gap in the training.

In this book we try to bridge this gap. The book starts at the undergraduate level when a student enters the MSc program and goes up to the tools needed for advanced courses.

Such a book will be useful to students, and to their teachers as well, for all the four semesters. It will be more like a *companion reader*, because different methods crop up in different courses, and there is no reason that a student has to learn all the methods in just the first semester.

Such a book can result in a massive, dull, and forbidding encyclopedia of a book, whereas what we are attempting is not a book for reference but a book primarily for inspiring students to learn and appreciate mathematical tools.

There are a great many, exhaustive, books in the tradition of Courant-Hilbert, Morse-Feshbach, B. Simon, W. Thirring, Choquet-Bruhat-DeWitt-Morette-Bleick or V. Balakrishnan, to name just the few most venerable authors. And our task is to prepare the student to approach those books with confidence.

An analogy might help to illustrate our purpose.

Being a good citizen does not mean having to read all the law books in order to be on the right side of law. There must be a minimum set of guidelines and public instructions to avoid going astray.

This book on mathematical physics will be such a set of instructions. And the book should be fun to read!

We know from our experience as teachers that many physics students, and even senior researchers, feel hesitation and discomfort when approaching mathematical topics. Our idea is to make them overcome that feeling. If this book succeeds in doing that, it would have served its purpose.

Pankaj Sharan and Ashok K. Kapoor

Part I: Basics

If people do not believe that mathematics is simple, it is only because they do not realize how complicated life is.

John von Neumann

Note to the reader: This part is not a complete or exhaustive summary of school level and undergraduate mathematics. It is meant for entertainment. If you are the serious type, you can directly go over to Part II.

Chapter 1

Logic and Set Theory

1.1 Statements and Truth values

Logic is the backbone of mathematics. The purpose of logic is to arrive at a true statement, starting from another statement whose truth is given.

Logic, and therefore also mathematics, deal with statements or **propositions**. All mathematics is expressed in propositions or statements which are either **true** or **false**. Statements like 'it might rain tomorrow' have no place in mathematics.

To every statement, let us denote it by P corresponds a statement which is exactly its opposite, or **negation**, denoted by $\sim P$. If P is true, then $\sim P$ is false, and if P is false, then $\sim P$ is true.

Simple statements can be combined in many ways to make **compound statements**.

There is **conjunction**, denoted by & and pronounced 'and'. Given P and Q, $P \& Q$ is true if both P and Q are separately true, in all other cases it is false.

Two statements P and Q can also be combined through **disjunction**, denoted by $\vee$ and pronounced 'or'. $P \vee Q$ is true if any one or both of these statements P, Q are true. It is false when both are false.

The ordinary language is not totally clear about the use of 'or' . Sometimes it is used in the sense that (P or Q) is true if only either of them is true, bot not both. And sometimes, it is true if both P and Q could also true. The logical circuit theory therefore uses 'XOR', the exclusive 'or', to remove this ambiguity.

1.2 Truth Tables

A **truth table** is to show the various truth values. For negation it is :

P	$\sim P$
T	F
F	T

where T and F stand for 'true' and 'false' respectively.

The **conjunction** of two sentences P and Q, $P\&Q$ denotes the ordinary meaning of 'and' :

P	Q	$P\&Q$
T	T	T
T	F	F
F	T	F
F	F	F

The **disjunction** of two sentences $P \vee Q$ (to be read as 'P or Q') is false when both P and Q are false, but is true in all other cases.

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

1.3 Implication

The **implication** $P \to Q$ denotes the relation 'if P then Q'. By definition its truth table is

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T

The sentence P is called the **antecedent** or **hypothesis** and Q the **consequent** or **conclusion**. Some explanation is necessary here :

The idea behind 'if P then Q' or $P \to Q$ is that P *implies* Q. That is, if it is known that P is true then we can validly *infer* that Q is true. This will make the compound sentence $P \to Q$ true. But if P is true and Q is found to be false then $P \to Q$ must be false.

But what happens when P is false?

There are no guarantees that Q will be true or false if the precondition P is not satisfied. Therefore, the definition of $P \to Q$ is given so that when P is false, $P \to Q$ is true regardless of the truth or falsehood of Q.

As an illustration, let P be the statement $1 = 0$ (which we know is false). And let Q be $3 = 7$ (false) and R be the statement $2 < 8$ (which is true). Then both $P \to Q$ and $P \to R$ are true:
If $1 = 0$ then $3 = 7$ (true)
If $1 = 0$ then $2 < 8$ (also true).

The sentence 'if ... then' is given the value true because we are safe in the knowledge that the antecedent is false anyway.

Beginners in logic get a feeling that unlike conjunction and disjunction the implication is not a compound sentence but a process of arriving at the truth of the conclusion given the truth of the hypothesis.

This feeling is not misplaced. *This actually is the purpose of implication.* If we find that $P \to Q$ is a **tautology**, that is, it is *always true* independently of the truth values of its components P, and Q, then we can draw the conclusion as follows :

	$P \to Q$	True always.
	P	True.
Therefore	Q is true	Because Q being false is not allowed.

A **biconditional** sentence or **equivalence** $P \leftrightarrow Q$ is just a shorter name for $(P \to Q)\&(Q \to P)$. Its truth table is

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

It is called equivalence because the antecedent and the consequent are logically equivalent. From the truth of P we can derive the truth of Q and from the truth of Q we can derive the truth of P.

The equivalence is *symmetric*, $P \leftrightarrow Q$ is the same as $Q \leftrightarrow P$.

1.4 Tautologies

A compound sentence which is always true irrespective of the truth or falsehood of its components is called a **tautology**.

For example, $(P \vee \sim P)$ is always true which just says either P is true or its opposite $\sim P$ is true. This tautology is called the *law of excluded middle*, because it means there is no common ground between P and its exact opposite $\sim P$. Equivalently, it says that 'it is always false that that P and its opposite $\sim P$ could both be true'. This later version is called the *law of contradiction*.

Some well known tautologies are given in the table below. By using the truth tables one can check that they are actually always true.

In Latin *tollens* means denying and *ponens* means asserting.

	Name	Tautology
1	Law of Excluded Middle	$P \vee \sim P$
2	Law of Contradiction	$\sim (P \& \sim P)$
	Implications	
3	Law of Detachment	$((P \to Q) \& P) \to Q$
4	*Modus tollendo tollens* or Law of Absurdity	$((P \to Q) \& \sim Q) \to \sim P$
5	*Modus tollendo ponens*	$((P \vee Q) \& \sim P) \to Q$
6	Law of Hypothetical Syllogism	$((P \to Q) \& (Q \to R)) \to (P \to R)$
7	Law of Exportation	$((P \& Q) \to R) \to (P \to (Q \to R))$
8	Law of Addition	$P \to (P \vee Q)$
	Equivalences	
9	Law of Double negation	$P \leftrightarrow (\sim\sim P)$
10	Law of Contraposition	$(P \to Q) \leftrightarrow (\sim Q \to \sim P)$
11	De Morgan's Laws	$[\sim (P \vee Q)] \leftrightarrow [\sim P \& \sim Q]$
12	De Morgan's Laws	$[\sim (P \& Q)] \leftrightarrow [\sim P \vee \sim Q]$
13	Equivalence of implication and disjunction	$(P \to Q) \leftrightarrow (\sim P \vee Q)$ $(P \vee Q) \leftrightarrow (\sim P \to Q) \leftrightarrow (\sim Q \to P)$

Example 1.1 Check the truth values of the tautology $((P \to Q)\&(Q \to R)) \to (P \to R)$.

Let us call $S = (P \to Q)\&(Q \to R)$. then the tautology is of the form $S \to (P \to R)$. Which is false only in the case when S is true and $(P \to R)$ is false. We should just check only this case to make sure that $S \to (P \to R)$ is a tautology.

P	Q	R	$P \to Q$	$Q \to R$	$S = (P \to Q)\&(Q \to R)$	$P \to R$
T	T	T	T	T	T	T
T	T	F	T	F	F	F
T	T	T	T	T	T	T
T	T	F	T	F	F	F
T	F	T	F	T	F	T
T	F	F	F	T	F	F
T	F	T	F	T	F	T
T	F	F	F	T	F	F

From the table it is clear that whenever $P \to R$ is false, so is S. Therefore $S \to (P \to R)$ is always true.

Exercise 1.1 Prove the other tautologies given in the table by constructing truth tables.

1.5 Getting familiar with logical inference

We give some examples below for familiarity with the logical inference.

Take the equivalence $(P \vee Q) \leftrightarrow (\sim P \to Q)$ which is actually the last (number 13). This is illustrated by the following example.

Example 1.2 In a certain university tuition fees are very high but the meritorious poor students are admitted and charged lower fees. Let
P= 'A student is meritorious.'
Q= 'A student is rich.'
For a general student $P \vee Q$ is true: either he (or she) is meritorious, or rich or both. But this is equivalent to $\sim P \to Q$ whose interpretation is 'if a student is not meritorious then he (or she) must be rich to get admitted'.

We also know that $P \vee Q$ is equivalent to $Q \vee P$ and so equivalent to $\sim Q \to P$ which means 'if a student is not rich then he (or she) must be meritorious'.

The main purpose of logic is to arrive at a conclusion Q when we are given that a number of premises $P_1, P_2, \ldots, P_n$ are true. This means that we must be able to show by some means or the other that the implication

$$(P_1 \& P_2 \& \cdots P_n) \to Q$$

is a tautology. There are other simpler means too as we shall see.

Example 1.3 Many university departments employ those students who get a scholarship as teaching assistants. Consider the following sentences :
(1) If a research scholar of the department does not get scholarship then she (or he) does not teach.
(2) If she (or he) does not teach then her (his) understanding does not improve.
(3) Either her (his) understanding improves or she (he) leaves research.
(4) But she (he) is not leaving research.
(5) Therefore, she (he) is getting scholarship.

Is the conclusion (5) valid?

To analyze this, use the abbreviations for a research scholar:
S= Gets scholarship
T= Teaches
U= Her (His) understanding improves
L= Leaves research.

The premises (1) to (4) and conclusion (5) can be written

P_1	$\sim S \to \sim T$
P_2	$\sim T \to \sim U$
P_3	$U \vee L$
P_4	$\sim L$
Q	S

The question therefore is : is $P_1 \& P_2 \& P_3 \& P_4 \& \to Q$ a tautology?

We can of course check all the 16 possibilities (2^4 truth values of 4 sentences S, T, U, L). But it is easy to proceed step by step. We note that $P_1 \& P_2$ implies

$\sim S \rightarrow \sim U$ (by the law of hypothetical syllogism). This is equivalent to $U \rightarrow S$ (by law of contraposition). Similarly P_3 is equivalent to $\sim U \rightarrow L$ (Equivalence of implication and disjunction) which is, in turn, equivalent to $\sim L \rightarrow U$ (by law of contraposition). Therefore, combining this with $P_4 = \sim L$ gives the conclusion S. Thus the conclusion is valid.

We summarize it as follows :

1	P_1	$\sim S \rightarrow \sim T$
2	P_2	$\sim T \rightarrow \sim U$
3	$1, 2$	$\sim S \rightarrow \sim U$
4	from 3	$U \rightarrow S$
5	P_3	$U \vee L$
6	from 5	$\sim U \rightarrow L$
7	from 6	$\sim L \rightarrow U$
8	P_4	$\sim L$
9	from 7,8	U
10	Therefore from 9, 4	S

Here is an economic-political argument from Patric Suppes' book.

Example 1.4 If either salaries are increased or prices go up then there is inflation. If there is inflation, then either the government controls inflation or people suffer. If people suffer then government becomes unpopular. But the government is not controlling inflation and the government is not unpopular.

Therefore salaries have not increased. (Yes or No?)

We analyze it as before by symbolizing :
S= salaries increase
Pi= prices increase
I= there is inflation
C= government controls
U= government is unpopular
Ps= people suffer

The premises are
$P_1 = (S \vee Pi) \rightarrow I$
$P_2 = I \rightarrow (C \vee Ps)$

$P_3 = Ps \to U$
$P_4 =\sim C\& \sim U$
and the expected conclusion is
$Q =\sim S$.

There are many ways to analyze it. One way is suggested here. The last premise P_4 is actually made up of two premises $P_5 =\sim C$ and $P_6 =\sim U$. P_6 and P_3 imply $\sim Ps$ therefore we have both $\sim Ps$ and $\sim C$. Now look at P_2 which involves both C and Ps. P_2 is equivalent to $\sim (C \vee Ps) \to\sim I$ and we know that $\sim (C \vee Ps)$ is the same as $\sim C\& \sim Ps$ thus we conclude $\sim I$. This combined with P_1 gives $\sim (S \vee Pi)$ which is the same as $\sim S\& \sim Pi$. Thus we not only get the conclusion $\sim S$ but as a byproduct (corollary) the conclusion that $\sim Pi =$ prices will not increase.

The law of absurdity occurs often in mathematics. It is often called 'reductio ad absurdum'. Very often in order to prove the truth of some statement, we assume that it is false. Then we are led to a contradiction, which is always false.

Chapter 2

Numbers

2.1 Natural Numbers and Integers

Quantitative sciences begin with counting. The numbers $1, 2, ...$ called **natural numbers** or whole numbers are the basis of all mathematics. These numbers have a natural order (like 5 is greater than 3 and 7 is less than 9 and so on).

Addition and multiplication are defined in the usual manner and rules, like the sum, or product of two numbers does not depend on the order in which they are added or multiplied: $5 + 3 = 3 + 5$ and $2 \times 7 = 7 \times 2$. Put in fancy words, addition and multiplication are **commutative**. Moreover, there is the **distributive law**, essentially stating that $2 \times (3 + 4) = 2 \times 3 + 2 \times 4$, and other similar statements.

If we multiply a number, say 5, with itself, it is called the square of the number, because if we lay down 5 rows of 5 objects each we get a picture of a **square**. Similarly **cube** and higher **powers** are defined as repeated multiplication of a number with itself. The powers are denoted by n^2 for square, n^3 for cube, and, more generally by n^m, denoting the multiplication of n with itself m times. It is called the m-th power of n. Powers of the same number obey the obvious but important rule:

$$n^r \times n^s = n^{r+s}.$$

Any set of objects is called **countable** if it can be put in one-to-one correspondence with natural numbers starting from 1 and increasing. The process of counting, that is, putting objects in one-to-one correspondence natural numbers, may end at some final number, then the set is called **finite** or counting may go on endlessly, then we say the set is **countably infinite**.

We can extend the set of natural numbers by adding 0 and negative whole numbers $-1, -2...$ etc. The process of addition and multiplication can be extended using the three unique numbers $-1, 1$ and 0 : if n is any natural number then

$$\begin{aligned} 0 + n &= n \\ 0 \times n &= 0, \\ (-1) + 1 &= 0, \\ (-1) \times (-1) &= 1, \\ (-1) \times n &= -n \end{aligned}$$

and so on. The set of all whole numbers, positive and negative, and the zero is called the set of **integers**.

It is conventional to omit the symbol for multiplication $\times$, or replace it with a dot or period . when there can be no confusion, such as in $1.2.3.4 = 1 \times 2 \times 3 \times 4$ and when using algebraic symbols such as $2x$ or xy.

Example 2.1 What is the sum of the first n natural numbers? One can guess the formula by observing the first few values of n:

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

To verify that it is indeed the correct formula we can calculate the sum of the first $n+1$ natural numbers by adding $(n+1)$ to this sum and check if we obtain the formula for $(n+1)$:

$$(1 + 2 + \cdots + n) + (n+1) = \frac{n(n+1)}{2} + (n+1) = \frac{(n+1)(n+2)}{2}.$$

The product of the first n natural numbers is called the **factorial** of n, denoted by $n!$ (the only place where exclamation mark appears in the serious business of mathematical discourse)

$$n! = 1 \times 2 \times \cdots \times n.$$

The factorial has the property that

$$n! = n \times (n-1)!.$$

As $1! = 1$, this property suggests that we choose, for convenience,

$$0! = 1.$$

Factorials appear naturally when we try to place n different objects in a single line in all possible ways, because we can place any of the n objects in the leftmost position, and for this choice we can place any of the remaining $(n-1)$ objects in the second position and so there are $n(n-1)$ possible ways to fill the first two positions. Similarly for each of these choices, the third position can be occupied by the remaining $(n-2)$ objects and so on till we reach the last position.

When we count the number of ways to choose $r < n$ objects out of n we encounter factorials even more commonly. Suppose we are required to find the number of ways to choose 3 objects out of 7, say. If we choose one object out of 7, we can do it in 7 different ways. For each of these 7 ways we can choose any of the remaining 6 objects. So there are 7×6 ways to choose two objects out of 7. When we choose the third object, the number of ways becomes $7 \times 6 \times 5$. But we are interested in the choise of three objects, not in any particular order in which they were drawn. There are 3! different order in which the same three objects can be chosen. We must divide the total number of ways by this: So the total number of ways in which 3 objects can be pulled out of 7 different objects is

$$\frac{7 \times 6 \times 5}{3!} = \frac{7!}{3!(7-3)!}.$$

The general formula for number of ways to choose r objects out of n is denoted by two popular symbols:

$${}^nC_r = \begin{pmatrix} n \\ r \end{pmatrix} = \frac{n!}{r!(n-r)!}$$

2.2 Rational Numbers

Next on the list is the set of **rational numbers** or **fractions**. They are defined via the process of **division**, which is the reverse of multiplication.

When we multiply 5 and 3, for example, we get the number 15. But it can only be divided by numbers 5 and 3, and not by, say, 4. Thus we define a new set of numbers like 15/4.

But where do we place 15/4 in the increasing order of numbers? If I were to divide 15 oranges (the favorite fruit of all mathematics school teachers!) to four children, each will get 3 each out of the 15 and, with the 3 remaining to be distributed among the four, each will get less than one orange. It is clear that the number 15/4 will be larger than 3 but less than 4. We write $15/4 = 3 + 3/4$, and 3/4 lies between 0 and 1.

Note that for any natural number n,

$$\frac{n}{n} = 1, \text{ and } \frac{n}{1} \quad = \quad n.$$

We can picture all numbers in a single line with larger numbers on the right of the smaller numbers. Fractions like 15/4 will fall between the whole numbers 3 and 4. What happens when two fractions are compared? Since a fraction can always be written as an integer (which can be zero) plus a fraction between 0 and 1, it is enough to compare only fractions within this range in order to settle the order of any two fractions. And here it is easy because $1/n$ is *less* than $1/m$ if n is *larger* than m. The negative fractions can be defined by multiplying -1 to them and extending the rules of addition, multiplication etc in the usual way. Thus the line of numbers can be filled up will all rational numbers, including the integers.

Example 2.2 How many fractions are there?

Are they countable? It may seem that rational numbers like $m/n, n \neq 0$ are many more than the natural numbers. But we can put the rationals in one-to-one correspondence with a set which has all the rational numbers, with many repetitions of the same rational number! In the following 'pyramid' the order of counting is shown by arrows line after line starting from the top.

$$\begin{array}{ccccccccccc}
 & & & & & 1/1 & & & & & \\
 & & & & 2/1 & \to & 1/2 & & & & \\
 & & & 3/1 & \to & 2/2 & \to & 1/3 & & & \\
 & & 4/1 & \to & 3/2 & \to & 2/3 & \to & 1/4 & & \\
 & 5/1 & \to & \dots & \to & \dots & \to & \dots & \to & 1/5 & \\
\dots & & & & & \dots & & & & & \dots
\end{array}$$

In this procedure of counting the fraction r/s appears as the s-th term on the $(r+s-1)$-th line. There are $1+2+\cdots+(r+s-2) = (r+s-2)(r+s-1)/2$ terms up to the $(r+s-2)$-th line, and then s more on the $(r+s-1)$-th line. That makes r/s in correspondence with the natural number

$$k = s + (r+s-2)(r+s-1)/2.$$

The upper (or first) integer in a fraction is called the **numerator** and the lower or (or second) part is called **denominator**. Fractions can be added if they have a common denominator:

$$\frac{m}{n} + \frac{k}{n} = \frac{m+k}{n}.$$

Fractions can be multiplied as follows:

$$\frac{n}{m} \times \frac{r}{s} = \frac{n \times r}{m \times s}$$

The multiplication rule allows us to add fractions with different denominators by suitable multiplication to the numerator as well as denominators by the same number (effectively multiplying by 1) so as to make the denominators common to both fractions:

$$\frac{3}{12} + \frac{7}{6} = \frac{3}{12} + \left(\frac{7}{6} \times \frac{2}{2}\right) = \frac{3}{12} + \frac{7 \times 2}{6 \times 2} = \frac{3+14}{12} = \frac{17}{12}.$$

We can extend the rule of addition of powers of a natural number to all integer powers. To begin with, the zeroth power of any natural number should be equal to 1, that is $n^0 = 1$ because

$$n^m \times n^0 = n^{m+0} = n^m.$$

Next, n^{-1} is equivalent to the fraction $1/n$ because

$$n \times n^{-1} = n^1 \times n^{-1} = n^0 = 1,$$

which is compatible with the multiplication rule if $n^{-1} = 1/n$. Similarly, $n^{-m} = 1/n^m$.

2.3 Irrational Numbers

We encounter another family of numbers called **irrational numbers**. Suppose we ask the question, "What is the number x which, when multiplied to itself gives the number 2 as a result?" The equation

$$x.x = x^2 = 2$$

defines a number $\sqrt{2}$ which is not a rational number. An irrational number like $\sqrt{2}$ lies on the border line of all those rational numbers whose squares are greater than and those whose squares are less than 2.

It is conventional to write $2^{1/2}$, the 'half' power, in place of $\sqrt{2}$, because $2^{1/2} \times 2^{1/2} = 2^{1/2+1/2} = 2^1 = 2$.

An equation like

$$x^2 - 2 = 0$$

is an example of an **algebraic equation** in which the unknown number occurs with integral powers and terms appear with rational coefficients.

2.4 Real numbers

There are irrational numbers which are not the solutions of algebraic equations. They are called **transcendental numbers**.

The set of all rational and irrational numbers constitute the set of all **real numbers** or the **real line**.

All real numbers can be obtained by a process of gradually better and better approximation of a **sequence** of rational numbers.

A sequence of rational numbers $a_1, a_2, \ldots, a_n, \ldots$ is called **convergent** or a **Cauchy sequence** if it has the property that the absolute difference between any two numbers of the sequence *sufficiently down the line* diminishes to zero. Put more precisely, it means this: if we choose a small rational number ϵ, then we can find a whole number N such that for $n > N$ and all $p > 0$

$$|a_n - a_{n+p}| < \epsilon.$$

Here we denote by $|a - b|$ the absolute value of the number: $|a - b| = a - b$ if $a > b$, and $|a - b| = b - a$.

> It is important to emphasize that once ϵ is chosen we have to find an n such that *for all possible choices* of p, however large, and including those depending on n, the difference $|a_n - a_{n+m}|$ can be made less than ϵ.

Two convergent sequences of rational numbers a_n and b_n are said to be *equivalent* if, given a small rational number ϵ we can find a sufficiently large number N such that

$$|a_n - b_n| < \epsilon, \ n > N.$$

The idea is that although the same real number can be defined by many convergent sequences, all these sequences must be equivalent in the sense defined above.

> *A real number is defined by any one of the equivalent Cauchy sequences, or as we say, by an* **equivalence class** *of Cauchy sequences of rational numbers.*

The earlier definition of a real number as lying on the borderline between two classes of rational numbers ($\sqrt{2}$ is sandwiched between rational numbers whose squares are greater than or less than 2) is equivalent to this. But we need not go into details of this.

From now on we use the intuitive notion of real numbers lying on the **real line**. The most useful property of real numbers is their being ordered: given any two real numbers a and b either $a < b$ or $a > b$, and if neither, they must be equal. Another property which we take for granted, and looks obvious, but is an independent axiom, is the **Archimedean property** of real numbers: given a positive number a and a positive number $b > a$, there exists a natural number N such that $Na > b$. This axiom rules out numbers called 'infinitesimals' because a small number can always be made larger than any number by multiplication by a large enough integer.

The simplest case of a sequence of positive rational numbers which approximates to a real number is a **strictly increasing** sequence of numbers all of which are also less than some positive number, or, as we say, **bounded from above**. Since the sequence is strictly increasing, and it cannot forever go up, it has to saturate to some value equal to or below the upper bound. Similarly, there is a case for a sequence which is **strictly decreasing** and **bounded from below**.

Example 2.3 **The base of natural logarithm** The number denoted by e (the base of natural or Napierian logarithm), is the limit of a sequence whose n-th entry is given by

$$r_n = \left(1 + \frac{1}{n}\right)^n .$$

These are all positive numbers. Moreover as

$$\begin{aligned}
\left(1 + \frac{1}{n}\right)^n &= 1 + 1 + \frac{n(n-1)}{1.2n^2} + \cdots + \frac{1}{n^n} \\
&= 2 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \\
&\quad + \cdots \frac{1}{n!}\left(1 - \frac{1}{n}\right)\cdots\left(1 - \frac{n-1}{n}\right)
\end{aligned}$$

a comparison of terms in r_{n+1} and r_n, shows that as

$$\frac{1}{p!}\left(1 - \frac{p}{n+1}\right) > \frac{1}{p!}\left(1 - \frac{p}{n}\right) \text{ for } p = 1, \ldots, n-1,$$

all terms of r_{n+1} are strictly greater than corresponding terms of r_n, and, in addition, there is the last positive term

$$\frac{1}{(n+1)!}\left(1 - \frac{1}{n+1}\right)\cdots\left(1 - \frac{n}{n+1}\right).$$

in r_{n+1}.

Since factors like $(1 - p/n)$ are strictly less than 1,

$$\begin{aligned} r_n &< 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} \\ &< 2 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \cdots + \left(\frac{1}{2}\right)^{n-1} \\ &= 2 + 1 - \left(\frac{1}{2}\right)^{n-1} \\ &< 3, \end{aligned}$$

showing that the sequence r_n is bounded from above. Therefore it converges to a number between 2 and 3.

Example 2.4 We have seen that the base of natural logarithm e can be approximated by the sequence

$$a_n = \left(1 + \frac{1}{n}\right)^n .$$

It is also approximated by equivalent and equally useful sequence,

$$b_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}.$$

Example 2.5 There is an old Babylonian recipe around 500 BCE, to calculate the square root of 2 by a sequence of fractions. It is approximated by the sequence $a_1, a_2, \ldots$ with a starting number a_1 chosen arbitrarily. The recipe is given by

$$a_{n+1} = \frac{a_n}{2} + \frac{1}{a_n}.$$

Suppose we calculate $\sqrt{2}$ starting with $a_1 = 1$, then we obtain the sequence,,

$$a_1 = 1,\ a_2 = \frac{3}{2},\ a_3 = \frac{17}{12},\ a_4 = \frac{577}{408},\ a_5 = \frac{665857}{470832}, \cdots$$

The last approximation to $\sqrt{2}$ by a_5 is correct to ten decimal places!

Exercise 2.1 The method for approximating the square root of 2 as given in the example above was re-discovered by Isaac Newton, and it is called the Newton-Raphson method in numerical analysis. The recipe to find the square root of a positive number A is

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{A}{a_n}\right).$$

Plot the graph $y = x^2 - A, A > 0$. The square-root $\pm\sqrt{A}$ are the x values where the graph cuts the x=axis (that is $y = 0$). Verify that the Babylonian-Newton-Raphson recipe is this:

(1) choose any non-zero initial value of x, say a_1

(2) find the corresponding point on the graph

(3) draw a tangent line from that point

(4) find where it meets the x-axis, let that point be $x = a_2$

(5) go to step (1) and repeat the steps with this as initial value.

Example 2.6 The simplest convergent sequence of rational numbers is $a_n = 1/n$. It is strictly decreasing

$$\frac{1}{n} - \frac{1}{n+p} = \frac{p}{n(n+p)} > 0 \text{ for all } n,$$

and converges to zero.

It is a Cauchy sequence because given a small positive rational number $1 > \epsilon > 0$, we can make

$$\frac{p}{n(n+p)} < \epsilon$$

by choosing $n > [1/\epsilon]$. Here [ ] represents the integer part of the number inside it.

2.5 Binomial Theorem

Suppose we take the n-th power of $(1 + x)$, where x is any real number,

$$(1 + x)^n = (1 + x) \times \cdots \times (1 + x).$$

The multiplication on the right hand side will be an expression with all powers of x from 0 to n. The 0-th power is just 1 taken from each of the factors; the first power will occur with a factor n because there are n ways choose x from one of the factors and 1 from the remaining $n - 1$. Similarly,

x^2 will occur as many times as we choose x from two factors and 1 from the remaining factors. Similarly for other powers. Thus

$$\begin{aligned}(1+x)^n &= 1+^nC_1x+\cdots+^nC_rx^r+\cdots+^nC_nx^n \\ &= 1+nx+\frac{n(n-1)}{2!}x^2+\cdots \\ &\quad +\frac{n(n-1)\cdots(n-r+1)}{r!}x^r+\cdots+x^n\end{aligned}$$

We shall see later that the binomial theorem holds for fractional powers as well. In that case the symbols ${}^nC_r$ have no meaning of course, but the expression in the second equality above makes sense. Also, in that case, the right hand side wlll not contain a finite number of terms, but an infinite series.

2.6 Limits

2.7 Infinite series

The sum of a sequence of real numbers like

$$S = a_0 + a_1 + \cdots + a_n + \cdots$$

has a meaning only if the **partial sums**

$$s_0 = a_0 \qquad s_1 = a_0 + a_1, \ldots \qquad s_n = a_0 + a_1 + \cdots + a_n, \qquad \ldots$$

are a convergent sequence. The limit of $s_1, s_2, \ldots, s_n, \ldots$ is called the sum of the infinite series.

A necessary consequence of a convergent series is that the higher terms of the series should ultimately go to zero: that is, $a_n \to 0$ as $n \to \infty$. *But this is not a sufficient condition*, as can be seen by the example of the **harmonic series**

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots.$$

Here $a_n = 1/n \to 0$ as $n \to \infty$, but

$$|s_n - s_{n+p}| = \frac{1}{n+1} + \cdots + \frac{1}{n+p}$$

cannot be made as small as possible for all values of p. We need only to choose $p = n$. then

$$\begin{aligned}
|s_n - s_{n+p}| &= \frac{1}{n+1} + \cdots + \frac{1}{2n}, \qquad (n \text{ terms}) \\
&> \frac{1}{2n} + \cdots + \frac{1}{2n} \\
&= \frac{1}{2}
\end{aligned}$$

Therefore the sequence s_n is not a convergent sequence and the harmonic series is not convergent to a sum. In fact, as all terms are positive, it diverges to $+\infty$.

Example 2.7 The **geometric series**

$$s = 1 + x + x^2 + \cdots x^n + \cdots$$

will converge only if $|x| < 1$ because the individual terms of the series x^n must go to zero as $n \to \infty$.

The partial sum of the first n terms is

$$s_n = 1 + x + x^2 + \cdots x^n = \frac{1 - x^{n+1}}{1 - x},$$

which follows easily from multiplying s_n by x and subtracting it from s_n. As $|x| < 1$, $x^{n+1} \to 0$, and the sum converges to

$$s = \frac{1}{1-x}, \qquad |x| < 1.$$

Chapter 3

Differential equations

Physics is the most basic of the sciences. There are physical quantities which are measured by experiments. We try to find relations between them. Every physical quantity, like the position of a particle, or the time, is represented by a **number** or a set of numbers. Relations between numbers are governed by **functions**, like the position of a particle as a function of time.

But that is not all.

The functions representing relationships satisfy **laws** which are mostly **differential equations**, like equation of motion in our example.

Why are differential equations so prevalent in physics?

Differential equations have the power to summarize an infinity of physical situations in a single formula.

That is what makes physics so fundamental, and the knowledge of mathematics so essential.

Example 3.1 A small body is dropped from a height, and the time taken for it to hit the ground is noted. The data for different bodies and heights for the time taken seems to fit the the formula

$$h = at^2. \tag{3.1}$$

This provides the differential equation (after differentiating twice),

$$\frac{d^2h}{dt^2} = 2a. \tag{3.2}$$

A general solution of this differential equation is

$$h = c + bt + at^2 \tag{3.3}$$

where b and c are arbitrary constants of integration.

The solutions to this equation contain more information than the data from which it was derived!

For example, it includes the cases when the body is thrown vertically upwards or downwards with some initial velocity, and the height is measured not necessarily form the point at which it was dropped.

Example 3.2 The differential equation which summarizes the family of all circles of unit radius in the x-y plane.

A circle with the center at (a, b) and unit radius consists of points (x, y) which are related by

$$(x - a)^2 + (y - b)^2 = 1. \tag{3.4}$$

Differentiating with respect to x, we get calling $y' = dy/dx$

$$2(x - a) + 2(y - b)y' = 0. \tag{3.5}$$

Differentiating it once more and denoting $y'' = d^2y/dx^2$,

$$1 + (y')^2 + (y - b)y'' = 0.. \tag{3.6}$$

We can eliminate a and b from the three equations (3.4), (3.5) and (3.6) and get the following differential equation

$$y'' = (1 + (y')^2)^{3/2}. \tag{3.7}$$

Exercise 3.1 (a) Solve the differential equation (3.7) and verify that its solutions indeed represent all circles with unit radius.
(b) What is the differential equation for the family of *all* circles with the center $(0, 0)$ at the origin and arbitrary radius?

Hint: (a) Solve in two steps. First solve the equation $du/dx = (1 + u^2)^{3/2}$ where $u = y'$. Then, once you have y' as a function of x, solve for y. (b) $dy/dx = -x/y$.

3.1 Simplest differential equations

The simplest differential equation is, of course nor different that just integrating a function:

$$\frac{dy}{dx} = f(x),$$

whose solution is

$$y = \int f(x)\,dx + c$$

where the first part is the indefinite integral of the function $f(x)$ and c is a constant, called the **constant of integration**. It occurs because on differentiating a constant gives zero. The ease or difficulty of integration depends how simple or difficult the function $f(x)$. The indefinite integrals of some simple functions are tabulated in the last section of this chapter.

3.2 Indefinite integrals of elementary functions

$f(x)$	$\int f(x)dx$
x^a $(a \neq -1)$	$x^{a+1}/a+1$
$1/x$	$\ln x$
e^{ax}	e^{ax}/a
a^x	$a^x/\ln a$
$\sin x$	$-\cos x$
$\cos x$	$\sin x$
$\tan x$	$-\ln\cos x$
$\cot x$	$\ln\sin x$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$1/(1+x^2)$	$\tan^{-1} x$
$1/(1-x^2)$	$(1/2)\ln[(1+x)/(1-x)]$
$1/\sqrt{1-x^2}$ $(x<1)$	$\sin^{-1} x$
$1/\sqrt{x^2-1}$ $(x>1)$	

Integrals of the type

$$\int \frac{1}{ax^2 + bx + c} dx$$

occur quite often and they can be brought to the form

$$\int \frac{1}{1 + y^2} dy$$

if

$$\Delta \equiv 4ac - b^2 > 0$$

or, to the form

$$\int \frac{1}{1 - y^2} dy$$

if $\Delta < 0$. The result is,

$$\begin{aligned} \int \frac{1}{X} dx &= \frac{2}{\sqrt{\Delta}} \tan^{-1} \frac{2ax + b}{\sqrt{\Delta}} \ (\Delta > 0) \\ \int \frac{1}{X} dx &= -\frac{1}{\sqrt{-\Delta}} \ln \left[\frac{2ax + b - \sqrt{-\Delta}}{2ax + b + \sqrt{-\Delta}} \right] \ (\Delta < 0) \end{aligned}$$

If $\Delta = 4ac - b^2 = 0$, then a and c have the same sign. By using $-X$ instead of X we can take both a and c as positive. Then $b = \pm 2\sqrt{ac}$ and so $X = (\sqrt{a}x \pm \sqrt{c})^2 = (ax + b/2)^2/a$. whose integral is

$$\int \frac{1}{X} = \frac{-1}{ax + b/2} \ .$$

3.3 Series solutions

The series solutions of $\sin x$ and $\cos x$ were obtained by Isaac Newton for approximating those functions using methods equivalent to differential equations. Newton had invented the calculus, so it should not be surprising.

If $\sin x$ is to be approximated by

$$y = \sin x = a_0 + a_1 x + a_2 x^2 + \cdots$$

it is clear from the geometric definition of sin that if x is expressed in radians, and if x is very small, $\sin x \approx x + \cdots$, so that $a_0 = 0$ and $a_1 = 1$. After two differentiations $y'' = -y$ and so

$$2a_2 + 3.2a_3x + 4.3a_4x^2 + = -[x + a_2x^2 + a_3x^3 + \cdots].$$

Comparing, coefficients of powers of x we see that

$$\begin{aligned} a_2 &= 0 \\ a_3 &= -\frac{1}{3.2} \\ a_4 &= -\frac{1}{4.3}a_2 \\ \cdots &= \cdots \\ a_n &= -\frac{1}{n(n-1)}a_{n-2} \\ \cdots &= \cdots. \end{aligned}$$

Thus all the even powers have zero coefficients and the odd powers survive with alternating signs and inverse of factorials

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots.$$

In a similar manner one can derive

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots.$$

A more interesting case is that of the binomial theorem for any power: If

$$y = (1+x)^r, \text{ where } r \text{ is a real number}$$

then

$$y' = r(1+x)^{r--1} = r\frac{y}{1+x}.$$

Expressing y as a power series, (the constant term has to be equal to 1 because at $x = 0$, $y = 1$),

$$y = 1 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots$$

$$
\begin{aligned}
(1+x)(a_1 + 2a_2 x + 3a_3 x^2 + \cdots + n a_n x^{n-1} + \cdots) &= \\
r(1 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots)
\end{aligned}
$$

Comparing coefficients of powers on the two sides

$$
\begin{aligned}
a_1 &= r \\
(2a_2 + a_1) &= r a_2 \\
\cdots &= \cdots \\
n a_n + (n-1) a_{n-1} &= r a_{n-1} \\
\cdots &= \cdots
\end{aligned}
$$

we obtain the standard formula

$$
\begin{aligned}
(1+x)^r &= 1 + rx + \frac{r(r-1)}{2!} x^2 + \cdots \\
&\quad + \frac{r(r-1)\cdots(r-p+1)}{p!} x^p + \cdots
\end{aligned}
$$