Mathematical Physics: Some Tricks

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1 Green's function for Poisson equation in $n > 2$ dimensions

The UGC-CSIR NET examination of December 2013 contained the following question:

The expression

$$\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2}+\frac{\partial^2}{\partial x_4^2}\right)\left(\frac{1}{x_1^2+x_2^2+x_3^2+x_4^2}\right)$$

is proportional to
(A)$\delta(x_1+x_2+x_3+x_4)$
(B) $\delta(x_1)\delta(x_2)\delta(x_3)\delta(x_4)$
(C) $(x_1^2+x_2^2+x_3^2+x_4^2)^{-3/2}$
(D) $(x_1^2+x_2^2+x_3^2+x_4^2)^{-2}$

The answer could be deduced from dimensional analysis alone. The given expression has dimension of L^{-4} if x_i have dimension L. Therefore only (B) or (D) could be correct because $\delta(x)$ has dimension of $1/x$. But the given expression is equal to zero for $x_1^2+x_2^2+x_3^2+x_4^2 \neq 0$ as can be seen by a short calculation. Therefore (D) is ruled out, and (B) is the correct answer.

But I thought we never tell our students about the Green's function of the Poisson equation in higher dimensions, even though it is just a step from 3-dimension. Also, doing the angular integral in 3-dimensional k-space explicitly is easy, I do not know how to do in it four dimensions. So I had to go to Gelfand-Shilov to check!

The following discussion is based on § 3.3, Chapter II of I. M. Gel'fand and G. E. Shilov, *Generalized Functions*, Volume I (Academic Press). This is the place I will turn to if I encounter any trouble with generalized functions! The student is supposed to know the definition of the Dirac delta and the Gamma functions. Just that.

1.1

The Poisson equation in n dimensions is

$$\nabla^2 \phi = \left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \right) \phi(\mathbf{x}) = \rho(\mathbf{x}). \tag{1}$$

Its Green function $G(\mathbf{x})$ is defined as

$$\left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \right) G(\mathbf{x}) = \delta(\mathbf{x}). \tag{2}$$

If we represent both the Green function as well as the Dirac delta function as Fourier transforms, then

$$G(\mathbf{x}) = \frac{1}{(2\pi)^n} \int g(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}\, d^n\mathbf{k}, \tag{3}$$

$$\delta(\mathbf{x}) = \frac{1}{(2\pi)^n} \int e^{i\mathbf{k}\cdot\mathbf{x}}\, d^n\mathbf{k}. \tag{4}$$

Substitution in the Poisson equation determines

$$g(\mathbf{k}) = -\frac{1}{k^2}, \qquad k^2 = |\mathbf{k}|^2, \tag{5}$$

and so,

$$G(\mathbf{x}) = -\frac{1}{(2\pi)^n} \int k^{-2} e^{i\mathbf{k}\cdot\mathbf{x}}\, d^n\mathbf{k}. \tag{6}$$

1.2

To find $G(\mathbf{x})$ we calculate the slightly more general integral, (at no extra cost)

$$F(\mathbf{x}) = \frac{1}{(2\pi)^n} \int k^{\lambda} e^{i\mathbf{k}\cdot\mathbf{x}}\, d^n\mathbf{k}, \qquad \lambda > -n. \tag{7}$$

The restriction on the real parameter λ has been made to avoid trouble at $k = 0$. In our case $\lambda = -2$ and $n > 2$, therefore $\lambda = -2 > -n$.

We first note the *scaling property* and *rotational invariance* of F.

If $a > 0$ then replacing $\mathbf{x}$ by $a\mathbf{x}$ we see that

$$F(a\mathbf{x}) = a^{-\lambda-n}F(\mathbf{x}), \tag{8}$$

because $\mathbf{k}\cdot a\mathbf{x} = a\mathbf{k}\cdot\mathbf{x}$ and we can change the variable of integration to $\mathbf{k}' = a\mathbf{k}$.

Similarly, if R is a rotation in the n dimensional space, the integral doesn't change because $\mathbf{k}\cdot R\mathbf{x} = R^{-1}\mathbf{k}\cdot\mathbf{x}$ and we can change the variable of integration to $\mathbf{k}' = R^{-1}\mathbf{k}$. Thus

$$F(R\mathbf{x}) = F(\mathbf{x}). \tag{9}$$

This tells us that F is a function $F(r)$ of $r = \sqrt{x_1^2 + \cdots + x_n^2}$ only and by using the scaling property $F(r) = r^{-\lambda-n}F(1)$. Thus,

$$F(\mathbf{x}) = Cr^{-\lambda-n}, \tag{10}$$

and the remaining effort now is to calculate the constant C which can only depend on n and λ.

1.3

The trick to evaluate C is to multiply both sides of (7) by $\exp(-x_1^2 - \cdots - x_n^2) = \exp(-r^2)$ and integrate over all $\mathbf{x}$ space. The left hand side is

$$\int F(\mathbf{x})e^{-r^2}d^n\mathbf{x} = C\int_0^\infty r^{-\lambda-n}e^{-r^2}r^{n-1}dr\,d\Omega_n$$

where

$$d^n\mathbf{x} = r^{n-1}dr\,d\Omega_n$$

and $d\Omega_n$ are the $(n-1)$-fold integrations over angular variables in n-dimensions. The angular integrations are trivially equal to Ω_n

the measure of surface of the unit sphere in n-dimensions. Thus, using the definition of the Gamma function,

$$\int F(\mathbf{x}) \exp(-x_1^2 - \cdots - x_n^2) d^n\mathbf{x} = \frac{1}{2} C \Omega_n \Gamma\left(-\frac{\lambda}{2}\right). \tag{11}$$

On the other hand, the right hand side is equal to

$$\frac{1}{(2\pi)^n} \int k^\lambda e^{i\mathbf{k}\cdot\mathbf{x} - x_1^2 - \cdots - x_n^2} \, d^n\mathbf{k} d^n\mathbf{x}.$$

Doing the $\mathbf{x}$ integrals, and using integrals like

$$\int e^{ik_1 x_1 - x_1^2} dx_1 = \sqrt{\pi} e^{-k_1^2/4}$$

we reduce the integration to

$$\frac{(\pi)^{n/2}}{(2\pi)^n} \int_0^\infty k^{\lambda+n-1} e^{-k^2/4} \, dk \, d\Omega_n = 2^{\lambda-1} (\pi)^{-n/2} \Omega_n \Gamma\left(\frac{\lambda+n}{2}\right) \tag{12}$$

Equating (11) and (12), we get the answer

$$C = 2^\lambda (\pi)^{-n/2} \frac{\Gamma\left(\frac{\lambda+n}{2}\right)}{\Gamma\left(-\frac{\lambda}{2}\right)},$$

and so,

$$\frac{1}{(2\pi)^n} \int k^\lambda e^{i\mathbf{k}\cdot\mathbf{x}} \, d^n\mathbf{k} = 2^\lambda (\pi)^{-n/2} \frac{\Gamma\left(\frac{\lambda+n}{2}\right)}{\Gamma\left(-\frac{\lambda}{2}\right)} r^{-n-\lambda}. \tag{13}$$

1.4

Coming back to our original problem in (6), for which $\lambda = -2$, we see that

$$G(\mathbf{x}) = -\frac{(\pi)^{-n/2}}{4} \Gamma\left(\frac{n}{2} - 1\right) r^{-n+2} \tag{14}$$

is the Green's function for Poisson equation in n-dimensions. For three dimensions it reduces to the familiar and important result

$$G(\mathbf{x}) = -\frac{1}{4\pi r},$$

and in four dimensions

$$G(\mathbf{x}) = -\frac{4}{\pi^2 r^2}.$$