Lessons in Quantum Field Theory

S Matrix in Interaction Picture

Quantized Schrodinger Field

A. K. Kapoor
http://0space.org/users/kapoor

akkapoor@cmi.ac.in; akkhcu@gmail.com

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§1 Overview

Objectives: To define S-matrix in interaction picture.

Prerequisites: Time development in interaction picture.

§2 Recall and Discuss

Expansion of fields

In the interaction picture the total Hamiltonian is split into two parts $H = H_0 + H'$. Let u_n denote the eigenfunctions of H_0 with eigenvalues E_n

$$H_0 u_n(\mathbf{x}) = E_n u_n(\mathbf{x}) \tag{1}$$

In the interaction picture the field operator obey EOM with H_0 as the Hamiltonian.

$$i\hbar\frac{d\psi(\mathbf{x},t)}{dt} = H_0\psi(\mathbf{x},t) \tag{2}$$

Taking the expansion of the field in terms of $u_n(x)$ as

$$\psi(\mathbf{x},t) = \sum_n a_n u_n(\mathbf{x}) e^{-iE_n t/\hbar}, \qquad \psi^\dagger(\mathbf{x},t) = \sum_n a_n^\dagger u_n^*(\mathbf{x}) e^{iE_n t/\hbar}, \tag{3}$$

We note that the operators a_n will be independent of time. Verify .

Commutation Relations

The field operators obey equal time commutation relations.

$$[\psi(\mathbf{x},t), \psi^\dagger(\mathbf{y},t)] = \delta(x-y) \tag{4}$$

The operators $a_m, a_n^\dagger$ obey commutation relations

$$[a_m, a_n^\dagger] = \delta_{mn} \tag{5}$$

Using these commutators, it is straight to verify

$$\psi(x) = \sum_n u_n(x,t) a_n, \quad \psi^\dagger(x) = \sum_n u_n^*(x,t) a_n^\dagger, \tag{6}$$

$$[a_n, \psi^\dagger(\mathbf{x},t)] = u_n^*, \qquad [\psi(\mathbf{x},t), a_n^\dagger] = u_n. \tag{7}$$

Multi particle states

The states corresponding to $\nu_1, \nu_2, ...$ particle in levels $m_1, m_2, ...$ are defined by

$$|\nu_1, \nu_2, ...\rangle = \prod_m \frac{(a_k^\dagger)^{\nu_k}}{\sqrt{\nu_k!}} |0\rangle. \tag{8}$$

§3 III Main Topics

§3.1 S matrix

S Matrix Make a hyperlink to a new node to

The time evolution of states in the interaction picture is given by

$$|\psi, t\rangle = U(t, t_0)|\psi, t_0\rangle, \tag{9}$$

where expression for $U(t, t_0)$ is the time ordered exponential

$$U(t, t_0) = T \exp\Big\{ -\frac{i}{\hbar}\int_{t_0}^{t} H'(\tau) d\tau \Big\} \tag{10}$$

Up to terms first order in H', we get

$$U(t, t_0) = I + \frac{1}{i\hbar}\int_{t_0}^{t_1} H'(\tau)\, d\tau. \tag{11}$$

The amplitude that a system known to in an initial state $|i\rangle$ at time t_1 will be found to be in final state $|f\rangle$ at time t_2 is given by

$$c_{fi} = \langle f|U(t_2, t_1)|i\rangle. \tag{12}$$

In practical application limits $t_1 \to -\infty, t_2 \to \infty$ are taken. This limit of the time evolution operator defines the S matrix

$$S = \lim_{t_1\to\infty, t_2\to\infty} U(t_2, t_1). \tag{13}$$

§3.2 The First Order Term

We shall be interested in computing the first order S- matrix elements

$$c_{fi}^{(1)} = {}_I\langle f|U(\infty, -\infty)|i\rangle_I = \frac{1}{i\hbar}\int_{t_0}^{t_1} {}_I\langle f|H_I'(\tau)|i\rangle_I\, d\tau. \tag{14}$$

The operator $H_I'(\tau)$ is the interaction picture operator. The right hand side the matrix element of interaction Hamiltonian in the interaction picture. This answer can be easily shown to be equal to the first order time dependent perturbation theory result obtained in the Schrodinger quantum mechanics.

$$c_{fi}^{(1)} = \frac{1}{i\hbar}\int_{t_0}^{t_1} {}_S\langle f|H_S'(\tau)|i\rangle_S\, e^{i\omega_{fi}\tau} d\tau. \qquad \text{Verify this}. \tag{15}$$

where $\omega_{fi} = \frac{E_f - E_i}{\hbar}$. Here the subscript S denotes the Schrodinger picture operators and states.