Lessons in Quantum Field Theory

Normal Products and Matrix Elements

Interaction Picture

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§1 Overview

We take up definition of normal product of operators and some examples of computation of matrix elements. We will work in the interaction picture.

Objectives To define normal product of operators and to give examples of computation of simple matrix elements.

§2 Recall and Discuss

Expansion of fields

In the interaction picture the total Hamiltonian is split into two parts $H = H_0 + H'$. Letu_n denote the eigenfunctions of H_0 with eigenvalues E_n

$$H_0 u_n(\mathbf{x}) = E_n u_n(\mathbf{x}) \tag{1}$$

Taking the expansion of the field in terms of $u_n(x)$ as

$$\psi(\mathbf{x}, t) = \sum_n a_n u_n(\mathbf{x}) e^{-iE_n t/\hbar}, \qquad \psi^\dagger(\mathbf{x}, t) = \sum_n a_n^\dagger u_n^*(\mathbf{x}) e^{iE_n t/\hbar}, \tag{2}$$

We note that the operators a_n will be independent of time.

Multi particle states

The states corresponding to $\nu_1, \nu_2, ...$ particle in levels $m_1, m_2, ...$ are defined by

$$|\nu_1, \nu_2, ...\rangle = \prod_m \frac{(a_k^\dagger)^{\nu_k}}{\sqrt{\nu_k!}} |0\rangle. \tag{3}$$

Commutation Relations

The field operators obey equal time commutation relations.

$$[\psi(\mathbf{x}, t), \psi^\dagger(\mathbf{y}, t)] = \delta(x - y) \tag{4}$$

The nonzero commutators of field with $a_n, a_n^\dagger$ can be worked out

$$[a_n, \psi^\dagger(\mathbf{x}, t)] = u_n^*(\mathbf{x}, t), \qquad [\psi(\mathbf{x}, t), a_n^\dagger] = u_n(\mathbf{x}, t) \tag{5}$$

where $u_n(\mathbf{x}, t) = u_n(\mathbf{x}) e^{-iE_n t/\hbar}$.

§3 Normal Product and Matrix Elements

§3.1 Normal product

Definition 1 *An expression is called normal form if it is a sum of product of operators such that in each term all the creation operator appear to the left of all annihilation operators.*

We use the notation $: A_1 A_2 A_3 \ldots A_n :$ to denote the products of the operators in the normal form.

Properties of normal product

[1] A normal product of a set of operators does not depend on the order of the operators. Thus
$$: A_1 A_2 A_3 \ldots :=: A_{i_1} A_{i_2} A_{i_3} \ldots A_{i_n} :$$
where $i_1, i_2, \ldots i_n$ is a permutation of $(1, 2, \ldots, n)$ An operator which is not in normal form can be expressed as a sum of terms which are in normal form and a c-number. As an example
$$a_n a_m^\dagger = a_m^\dagger a_n + \delta_{mn}. \tag{6}$$
The right hand side is normal form of the product $a_n a_m^\dagger$.

[2] The vacuum expectation value of a normal product of operators is always zero.
$$\langle 0| : A_1 A_2 A_3 \ldots A_n : |0\rangle = 0. \tag{7}$$

[3] The vacuum expectation value of a normal product is zero.However, it will have nonzero matrix element between some states. For example
$$\langle \Psi| : a_{m_1}^{\dagger p_1} a_{m_2}^{\dagger p_2} \ldots\ldots a_{n_1}^{q_1} a_{n_2}^{q_2} \ldots : |\Phi\rangle \tag{8}$$
will be nonzero if the states $|\Psi\rangle, |\Phi\rangle$ meet the following requirements.

$|\Psi\rangle \longrightarrow p_1$ particles in state m_1, p_2 particles in state m_2
$|\Phi\rangle \longrightarrow q_1$ particles in state m_1 q_2 particles in state m_2, etc.

§3.2 Examples

We take up some examples showing how to
(i) order a product into a normal form, and
(ii) compute matrix elements by bringing operators into a normal form.
We use notation $|m\rangle$ to denote state of single particle in level m, and $|m, n\rangle$ denotes two particle state with one particle in levels m and n each. In our notation states with multi particles in a level are denoted by $|\nu_n, ..\rangle$ etc.

Example 1 Recall that
$$\psi(\mathbf{x}, t) = \sum_n u_n(\mathbf{x}, t) a_n, \qquad \psi^\dagger(\mathbf{x}, t) = \sum_n u_n^*(\mathbf{x}, t) a_n^\dagger. \tag{9}$$

While $\psi^\dagger(\mathbf{x}, t)\psi(\mathbf{x}, t)$ is in normal form, $\psi(\mathbf{x}, t)\psi^\dagger(\mathbf{x}, t)$ is not in a normal form. We now take up examples of computing matrix elements. This skill will be required for computing life times and cross sections.

Example 2 As a next example, we will arrange $\psi(\mathbf{x}_1,t)\psi(\mathbf{x}_2,t)\psi^\dagger(\mathbf{x}_3,t)$ in the normal form. We need to push $\psi^\dagger$ to the left most position.

$$\begin{aligned}
&\psi(\mathbf{x}_1,t)\psi(\mathbf{x}_2,t)\psi^\dagger(\mathbf{x}_3,t) \\
&= \psi(\mathbf{x}_1,t)\Big\{\psi^\dagger(\mathbf{x}_3,t)\psi(\mathbf{x}_2,t)+\big[\psi(\mathbf{x}_2,t),\psi^\dagger(\mathbf{x}_3,t)\big]\Big\} \quad (10)\\
&= \psi(\mathbf{x}_1,t)\psi^\dagger(\mathbf{x}_3,t)\psi(\mathbf{x}_2,t)+\psi(\mathbf{x}_1,t)\delta(\mathbf{x}_2-\mathbf{x}_3)\\
&= \Big\{\psi^\dagger(\mathbf{x}_3,t)\psi(\mathbf{x}_1,t)+\big[\psi(\mathbf{x}_1,t),\psi^\dagger(\mathbf{x}_3,t)\big]\Big\}\psi(\mathbf{x}_2,t)+\delta(\mathbf{x}_1-\mathbf{x}_2)\psi(\mathbf{x}_1,t)\\
&= \psi^\dagger(\mathbf{x}_3,t)\psi(\mathbf{x}_1,t)\psi(\mathbf{x}_2,t)+\delta(\mathbf{x}_1-\mathbf{x}_3)\psi(\mathbf{x}_2,t)+\delta(\mathbf{x}_1-\mathbf{x}_2)\psi(\mathbf{x}_1,t) \quad (11)\\
&= :\psi^\dagger(\mathbf{x}_3,t)\psi(\mathbf{x}_1,t)\psi(\mathbf{x}_2,t):+\delta(\mathbf{x}_1-\mathbf{x}_3)\psi(\mathbf{x}_2,t)+\delta(\mathbf{x}_1-\mathbf{x}_2)\psi(\mathbf{x}_1,t) \quad (12)
\end{aligned}$$

Note that the first term in (11) is, by definition, equal to the normal ordered form.

Example 3 In this example will calculate the matrix elements $\langle 0|\psi(\mathbf{x},t)|n\rangle$ and $\langle n|\psi^\dagger(\mathbf{x},t)|0\rangle$.
As a preparation, let us first compute the commutator

$$\begin{aligned}
\big[\psi(\mathbf{x},t),a_n^\dagger\big] &= \Big[\sum_j u_j(\mathbf{x},t)a_j,a_n^\dagger\Big]=\sum_j u_j(\mathbf{x},t)[a_j,a_n^\dagger] \quad (13)\\
&= \sum_j u_j(\mathbf{x},t)\delta_{jn}=u_n(\mathbf{x},t) \quad (14)
\end{aligned}$$

We shall use the identity $AB=[A,B]+BA$ repeatedly.

$$\begin{aligned}
\langle 0|\psi(\mathbf{x},t)|n\rangle &= \langle 0|\psi(\mathbf{x},t)\,a_n^\dagger|0\rangle \quad (15)\\
&= \langle 0|[\psi(\mathbf{x},t),a_n^\dagger]+a_n^\dagger\psi(x,t)|0\rangle \quad (16)\\
&= \langle 0|u_n(\mathbf{x},t)|0\rangle \qquad \because \langle 0|\,a_n^\dagger=0 \quad (17)\\
\therefore\quad \langle 0|\psi(\mathbf{x},t)|n\rangle &= u_n(\mathbf{x},t). \quad (18)
\end{aligned}$$

In a similar fashion we can get

$$\begin{aligned}
\langle n|\psi^\dagger(\mathbf{x},t)|0\rangle &= \langle 0|a_n\,\psi^\dagger(\mathbf{x},t)|0\rangle \quad (19)\\
&= \langle 0|[a_n,\psi^\dagger(\mathbf{x},t)]+\psi^\dagger(x,t)a_n|0\rangle \quad (20)\\
&= \langle 0|u_n(\mathbf{x},t)|0\rangle \qquad \because a_n|0\rangle=0 \quad (21)\\
\therefore\quad \langle 0|\psi^\dagger(\mathbf{x},t)|n\rangle &= u_n^*(\mathbf{x},t). \quad (22)
\end{aligned}$$

Example 3 Using $AB=[A,B]+BA$ repeatedly to shift annihilation operators to the right and creation operators to the left, we compute

$$\begin{aligned}
&\langle m|\psi^\dagger(\mathbf{x},t)\psi(\mathbf{y},t)|n\rangle \quad (23)\\
&= \langle 0|a_m\psi(\mathbf{x},t)\ \psi(\mathbf{y},t)a_n^\dagger|0\rangle \quad (24)\\
&= \langle 0|\Big([a_m,\psi^\dagger(\mathbf{x},t)]+\psi^\dagger(\mathbf{x},t)a_m\Big)\times\Big([\psi(\mathbf{y},t),a_n^\dagger]+a_n^\dagger\psi(\mathbf{y},t)\Big)|0\rangle \quad (25)\\
&= \langle 0|\Big(u_m^*(\mathbf{x},t)+\psi(\mathbf{x},t)a_m\Big)\times\Big(u_n(\mathbf{y},t)+a_n^\dagger\psi(\mathbf{y},t)\Big)|0\rangle \quad (26)\\
&\qquad \because a_m|0\rangle=0 \qquad \langle 0|a_n^\dagger=0 \quad \text{and} \quad (27)
\end{aligned}$$

Therefore only one of the four terms in (27) survives and we get

$$\langle m|\psi^\dagger(\mathbf{x},t)\psi(\mathbf{y},t)|n\rangle=u_m^*(\mathbf{x},t)u_n(\mathbf{y},t) \quad (28)$$

Example 4 Using the same strategy as above it can be proved that

$$\langle 0|\psi(\mathbf{x},t)\psi(\mathbf{y},t)|m,n\rangle = [u_m(\mathbf{x},t)u_n(\mathbf{y},t) + u_n(\mathbf{x},t)u_m(\mathbf{y},t)] \quad \text{Verify} \tag{29}$$

Notice how wave functions appear in the matrix elements of field operators. Apart from normalization factor $1/\sqrt{2}$, the right hand side is just the wave function for state $|m,n\rangle$.

§4 EndNotes

The examples here are meant to illustrate the normal form and computation of matrix elements. Eventually. Wick's theorem and a few standard rules for handling expressions, like those in above examples, will enable us to write the answers directly.