Problem Solving Session in Quantum Field Theory

Potential Scattering Cross Section

IInd Quantized Schrodinger Field

A. K. Kapoor
http://0space.org/users/kapoor
akkapoor@cmi.ac.in; akkhcu@gmail.com

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§1 Session Overview

Goals: To compute scattering cross section for particle; Second quantized Schrodinger field in external field (potential) $V(x)$.

To learn application of first order perturbation theory.

Skills needed The skill to compute matrix elements of product of field operators between Fock states; Properties of normal product

§2 Statement of the Problem

Assume that the interaction of Schrodinger field with an external source is given by

$$\mathscr{H}'(t) = \int d\mathbf{x}\psi^\dagger(\mathbf{x},t)V(\mathbf{x})\psi(\mathbf{x},t)$$

Show that differential scattering cross section is give by

$$\sigma(\theta) = |f(\theta,\phi)|^2$$

where f is the scattering amplitude

$$f(\theta,\phi) = \frac{m}{2\pi\hbar^2}\int d\mathbf{x}e^{i(\mathbf{k}_i-\mathbf{k}_f)\cdot\mathbf{x}}V(\mathbf{x})$$

§3 Recall formulae needed

Expansion of fields

In all the examples here we work in the interaction picture. The field operator will be expanded as

$$\psi(\mathbf{x}, t) = \sum_n a_n u_n(\mathbf{x}, t) \tag{1}$$

where $\{u_n(\mathbf{x}, t), n = 1, 2, ..\}$ are stationary solutions for Hamiltonian H_0

$$u_n(\mathbf{x}, t) = u_n(\mathbf{x}) e^{-iE_n t/\hbar} \tag{2}$$

First order perturbation result

$$\text{Transition amplitude } c_{fi} = -\frac{1}{i\hbar} \int_{t_1}^{t_2} \langle f | H_I'(t) | i \rangle dt$$

Some Commutators

$$\begin{aligned} [a, \psi^\dagger(x, t)] &= u^*_{\mathbf{k}}(\mathbf{x}) \Rightarrow a\psi^\dagger(\mathbf{x}) = u^*_{\mathbf{k}}(\mathbf{x}) + \psi^\dagger(\mathbf{x}) a & (3) \\ [\psi(x, t), a^\dagger] &= u_{\mathbf{k}}(\mathbf{x}) \Rightarrow \psi(\mathbf{x}) a^\dagger = u_{\mathbf{k}}(\mathbf{x}) + a^\dagger \psi(\mathbf{x}) & (4) \end{aligned}$$

The initial and final states

Initial and final states have one particle. Corresponding wave functions are

$$u_i(\mathbf{x}) = \frac{e^{\mathbf{k}_i \dot{\mathbf{x}}}}{(2\pi)^{3/2}}, \quad u_f(\mathbf{x}) = \frac{e^{\mathbf{k}_f \dot{\mathbf{x}}}}{(2\pi)^{3/2}}.$$

The final states are the states with energy E_f and momentum $\mathbf{k}_f$, with $E_f = \frac{\hbar^2 k_f^2}{2m}$, and momentum vector in the solid angle $d\Omega$.

Cross section

The differential cross section for potential scattering is given by

$$\sigma(\theta, \phi) d\Omega = \frac{w_{fi}}{J_{\text{inc}}} \tag{5}$$

where w_{fi} is the transition probability per unit time for scattering into solid angle $d\Omega$.
$\vec{J}_{\text{inc}}$ is calculated from probability current density for wave function $u_{\mathbf{k}i}(x)$ and is given by

$$\vec{J}_{\text{inc}} = \frac{1}{(2\pi)^{3/2}} \frac{\hbar \mathbf{k}_i}{m}. \tag{6}$$

Setting up the problem

Transition amplitude

The transition amplitude is given by

$$c_{\mathrm{fi}} = \frac{1}{i\hbar}\int_0^t \langle f|H'|i\rangle d\mathbf{x} \tag{7}$$

$$= \frac{1}{i\hbar}\langle f|H'|i\rangle \int_0^t e^{i(E_f - E_i)t/\hbar}\, dt \tag{8}$$

$$= \frac{1}{i\hbar}\langle f|H'|i\rangle \frac{2\sin(\omega_{fi}t/2)}{\omega_{fi}}^{i\omega_{fi}t/2} \quad \text{Verify} \tag{9}$$

where $\omega_{fi} = \dfrac{E_f - E_i}{\hbar}$.

Probability per unit time

The probability is given by $|c_{\text{fi}}|^2$, and the probability per unit time is

$$
\begin{aligned}
w_{\text{fi}} &= \frac{d}{dt}\left[\frac{1}{\hbar^2}\left|\langle f|V(\mathbf{x})|i\rangle\right|^2 \frac{4\sin^2(\omega_{fi}t/2)}{\omega_{fi}^2}\right] && (10)\\
&= \frac{1}{\hbar^2}\left|\langle f|V(\mathbf{x})|i\rangle\right|^2 \times 2\frac{\sin\omega_{fi}t}{\omega_{fi}} && (11)\\
&= \frac{2}{\hbar^2}\frac{\sin\omega_{fi}t}{\omega_{fi}} \times |V_{fi}|^2 && (12)
\end{aligned}
$$

where we $V_{fi}$is the matrix element

$$V_f i = \langle f|V(\mathbf{x})|i\rangle = \frac{1}{(2\pi)^3}\int d\mathbf{x} e^{i(\mathbf{k}_i-\mathbf{k}_f)\cdot\mathbf{x}} V(\mathbf{x})\, d\mathbf{x}. \tag{13}$$

Taking the limit $t \to \infty$, and using

$$\lim_{t\to\infty}\frac{\sin\omega_{fi}t}{\omega_{fi}} = \pi\delta(\omega_{fi}) \tag{14}$$

we get the transition probability per unit time equal to

$$w_{\text{fi}} = \frac{2\pi}{\hbar}|V_{fi}|^2\delta(\omega_{fi}). \tag{15}$$

Sum over final states

For differential cross section the final states of interest are those having final momentum direction in solid angle $d\omega$, summing over the final states will be carried over as

$$\sum_{\text{final}} (.) = d\Omega \int k^2 dk(.) \tag{16}$$

and k_f integral can be carried out using Dirac delta function

$$\delta(\omega_{fi}) = \delta\Big(\frac{E_f - E_i}{\hbar}\Big) = \hbar\delta(E_f - E_i).$$

To do this, change the integration variable to $E_f = \frac{\hbar^2 k_f^2}{2m}$ and make use of

$$dE_f = \frac{\hbar^2 k_f d\mathbf{k}_f}{m} \to k_f^2 dk_f = dE_f \frac{m k_f}{\hbar^2}$$

Hence we have

$$\sum_{\text{final}} (.) \;=\; d\Omega \int k^2 dk \delta(E_f - E_i)|V_{fi}|^2 \tag{17}$$

$$\tag{18}$$

The delta function $\delta(E_f - E_i)$ requires $E_f = E_i \to k_f = k_i$

$$\begin{aligned}
&\text{Prob per unit time for momentum in range } d\Omega && (19)\\
&= \frac{2\pi}{\hbar} \sum_{\text{final state}} w_{\text{fi}} && (20)\\
&= \frac{2\pi}{\hbar} d\Omega \int k_f^2 dk_f \delta\omega_{fi} \times |V_{fi}|^2 && (21)\\
&= \frac{2\pi}{\hbar} d\Omega \int dE_f \frac{m}{\hbar^2} k_i^2 dk_f \delta\omega_{fi} |V_{fi}|^2 && (22)\\
&= \frac{2\pi}{\hbar} (d\Omega) \Big(\frac{mk_i}{\hbar^2}\Big) |V_{fi}|^2 && (23)
\end{aligned}$$

Now compute cross section

By definition of differential cross section, the expression (20) is equal to $|J_{\rm inc}|\sigma(\theta,\phi)d\Omega$. Using $|J_{\rm inc}| = \dfrac{\hbar k_i}{(2\pi)^3 m}$ gives us

$$\begin{aligned}\sigma(\theta,\phi) &= \frac{1}{|J_{\rm inc}|}\times\frac{2\pi}{\hbar}\sum_{\text{final state}} w_{\rm fi} && (24)\\ &= \frac{(2\pi)^3 m}{\hbar k_i}\times\frac{2\pi}{\hbar}\times\Big(\frac{mk_i}{\hbar^2}\Big)\times|V_{fi}|^2 && (25)\\ &= \Big(\frac{m}{2\pi\hbar^2}\Big)^2\Big|\int d\mathbf{x}e^{i(\mathbf{k}_i-\mathbf{k}_f)\cdot\mathbf{x}}V(\mathbf{x})\,d\mathbf{x}\Big|^2 && (26)\end{aligned}$$

This agrees with the Born approximation result from time independent approach to potential scattering in quantum mechanics.

§4 Useful Tips

It will be seen that squaring gives

$$(2\pi)\delta(\omega_{fi}) \xrightarrow{\text{square}} (2\pi)^2\delta(0)\delta(\omega_{fi})$$

and computing transition probability per unit volume amounts to dropping the factor $(2\pi)\delta(0)$:

$$\boxed{(2\pi)\delta(\omega_{fi}) \overbrace{\longrightarrow}^{\text{square}} (2\pi)^2\delta(\omega_{fi})\delta(\omega_{fi}) = (2\pi)^2\delta(0)\delta(\omega_{fi}) \overbrace{\qquad\longrightarrow\qquad}^{\text{per unit time}} (2\pi)\delta(\omega_{fi})}$$

This replacement can be intuitively understood as follows

$$\begin{aligned}(2\pi)\delta(\omega) &= \int_{-\infty}^{\infty} e^{i\omega t}\,dt && (27)\\ &= \lim_{T\to\infty}\int_{T/2}^{T/2} e^{i\omega t}\,dt && (28)\\ (2\pi)\delta(0) &= \lim_{T\to\infty}\int_{T/2}^{T/2} e^{i\omega t}\,dt\Big|_{\omega=0} && (29)\\ \int_{T/2}^{T/2} e^{i\omega t}\,dt\Big|_{\omega=0} &= \int_{T/2}^{T/2} dt && (30)\\ &= T. && (31)\end{aligned}$$

Hence taking per unit time amounts to replacement

$$(2\pi)\delta(\omega)\Big|_{\omega=0} \to 1$$

] ≎ The above sequence of statement cannot be justified, a complete argument has however been given earlier.