Lectures in Quantum Field Theory
Bosons and Fermions

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Contents

§1 Lesson Overview

Syllabus

Connection between second quantized Schrodinger theory and nonrelativistic quantum mechanics. Hilbert space of second quantized theory;

Prerequisites

Quantum mechanics of identical particles; Symmetrization postulate; Spin statistics connection; Bosons and fermions.

Objectives

1. To recall and discuss quantum mechanics of identical particles; Symmetrization postulate.
2. To show and establish connection of states in second quantized theory and wave function in quantum mechanics. To demonstrate that CCR quantization leads to correctly symmetrized wave function for a two particle state.
3. To briefly describe the connection between description of assembly of many identical bosons in quantum mechanics and Hilbert space of the second quantized Schrodinger theory. To describe briefly the changes needed in quantization of Schrodinger field to describe system of identical fermions.
4. To give a brief about spin statistics connection.

§2 Recall and Discuss

§2.1 QM of Identical Particles

In nonrelativistic quantum mechanics, when describing a system of identical particles, the total wave function must be symmetrized for bosons and antisymmetrized for fermions. This is known as **symmetrization postulates** and has to be put in "by hand" in nonrelativistic theory.

The symmetrization (or antisymmetrization as the case may be) is to be insisted under exchange of all space as well as spin variables.

§3 Connection with Quantum Mechanics

§3.1 Schrodinger field for a particle with spin

To describe a spin s particle, we need to introduce a $(2s+1)$ component field $\psi_\alpha(\mathbf{x}), \alpha = -s, -s+1, ..., s$.

To simplify our notation we consider spin zero particle and show that quantization using commutators leads to correctly symmetrized wave functions.

As before, we choose a set of basis functions $u_n(x)$. Let $\psi(\mathbf{x})$ be expanded in a set of complete orthonormal functions$\{u_m(x)\}$:

$$\psi(\mathbf{x}) = \sum_n a_n u_n(\mathbf{x}). \tag{1}$$

with the operators $a_m, a_m^\dagger$ given by

$$a_m = \int d\mathbf{x} u_m^*(\mathbf{x})\psi(\mathbf{x}), \quad a_m^\dagger = \int d\mathbf{x} u_m(\mathbf{x})\psi^\dagger(\mathbf{x}) \tag{2}$$

For a single particle state $|m\rangle = a_m^\dagger|0\rangle$

$$\langle 0|\psi(\mathbf{x})|m\rangle = u_m(\mathbf{x}) \tag{3}$$

is just the Schrodinger wave function.

For a two particle state $|m, n\rangle = a_m^\dagger a_n^\dagger|0\rangle$, a straight forward computation shows that

$$\langle 0|\psi(\mathbf{x}_1)\psi(\mathbf{x}_2)|mn\rangle = \frac{1}{\sqrt{2}}\big(u_m(\mathbf{x}_1)u_n(\mathbf{x}_2) + u_m(\mathbf{x}_2)u_n(\mathbf{x}_1)\big). \tag{4}$$

Thus we get a properly symmetrized two particle wave function. In a similar manner it is possible to show that the wave function for any number of particles automatically comes out to be properly symmetrized.

§3.2 Many particle system in quantum mechanics

Let x collectively denote the space variables $\mathbf{x}$ and the spin variable α.

In Schrodinger quantum mechanics of Nidentical particles, one needs to work with wave function, $\psi(x_1, x_2, ..., x_N)$ depending in N position coordinates obeying the Schrodinger equation of the form

$$i\hbar\frac{d}{dt}\psi(x_1, x_2, ..., t) = -\frac{\hbar^2}{2m}\sum_{k=1}^{N}\nabla_k^2\psi(x_1, x_2, ..., x_N, t) + V(x_1, x_2, ...)\psi(x_1, x_2, ..., x_N, t) \tag{5}$$

The Hilbert space of states of one particle consists of square integrable functions of one coordinate set, $\sum_{\text{spins}}\int d\mathbf{x}|\psi(x)|^2 < \infty$. For two particles it is set of all symmetric functions $\sum_{\text{spins}}\int d\mathbf{x}_1 d\mathbf{x}_2|\psi(x_1, x_2)|^2 < \infty$, and so on

The following table displays the form of wave function, a basis, Hilbert space of states for quantum mechanical description of an assembly of several identical bosons.

	Wave function	Basis	Hilbert space
1	$\psi(x)$	$u_n(x)$	$\mathcal{H}$
2	$\psi(x_1, x_2)$	$\frac{1}{2}[u_{n_1}(x_1)u_{n_2}(x_2) + x_1 \leftrightarrow x_2]$	$\mathcal{S}(\mathcal{H}\otimes\mathcal{H})$
3	$\psi(x_1, x_2, x_3)$	$\frac{1}{3!}\sum_{\text{Perm}} u_{n_1}(x_1)u_{n_2}(x_2)u_{n_3}(x_3)$	$\mathcal{S}(\mathcal{H}\otimes\mathcal{H}\otimes\mathcal{H})$
....	...	...	...
N	$\psi(x_1, x_2, \ldots, x_N)$	$\frac{1}{n!}\sum_{\text{Perm}} u_{n_1}(x_1)\ldots u_N(x_N)$	$\mathcal{S}\left(\mathcal{H}\otimes\mathcal{H}\otimes\ldots\otimes\mathcal{H}\right)$

where $\mathcal{S}$ stands for symmeteric part of the tensor product.

Schrodinger QM vs second quantized theory

1. The entries in the last column, for $N > 1$ have symmetric tensor product of one particle Hilbert space $\mathcal{H}$

2. In contrast, in the second quantized theory one works with field operator $psi(\mathbf{x})$ to describe system with any number of identical particles.

3. The symmetrization requirement is automatically taken care off by making choice of appropriate CCR or CAR.

4. The Hilbert space for the second quantized theory is the direct sum of symmetric tensor products

$$\mathcal{H} \oplus \mathcal{S}(\mathcal{H} \otimes \mathcal{H}) \oplus \ldots \oplus \mathcal{S}(\mathcal{H} \otimes \mathcal{H} \ldots) \ldots \tag{6}$$

for all values of N.

Fermions An alternative quantisation procedure to quantise the Schrödinger equation is to assume that equal time anticommutator is given by $i\hbar$ times the Poisson bracket:

$$\big[\psi(x,t), \pi(y,t)\big]_{+} = i\hbar\delta(x-y), \tag{7}$$

where $[A,B]_{+} = AB + BA$ is the anticommutator of operators A, B.

1. The creation and annihilation operators satisfy the relations

$$[a_k, a_\ell^\dagger]_+ = \delta_{k\ell}, \quad [a_k, a_\ell]_+ = 0 \quad [a_k^\dagger, a_\ell^\dagger]_+ = 0 a_k^2 = 0, \qquad a_k^{\dagger 2} = 0 \tag{8}$$

(9)

2. For the number operators We have $N_k^2 = N_k$. This implies that the eigenvalues of number operators are 0 and 1 only. As result a level can be occupied by at most one particle.

3. $a_k^2 = 0, a_k^{\dagger^2} = 0$, If we apply a_k more than once, on any state we get null vector:

$$(a_k^{\dagger 2})|\nu_1, \nu_2, ...\rangle = 0$$

This means that it is not possible to create more than one particle in any given state.

To summarize, we have the result that use of anticommutators to quantize the Schrodinger equation, we would get a consistent description of identical fermions.

The Hilbert space of the second quantized theory for fermions consists of direct sum of anti-symmetric tensor products, instead of symmetric tensor products.

§4 Spin statistics connection

In nonrelativistic quantum theory the spin symmetrization postulate is an extra assumption. It is derived from experimental observations and leads to Pauli exclusion principle for fermions.

The spin statistics connection states that the relativistic field for integral spins must be quantized using CCR and field corresponding to half integral spin must be quantized with CAR.

The spin statistics connection is a consequence of very general requirements of relativistic invariance, causality and positive definiteness of energy.

§5 EndNotes

§5.1 About second quantization

... The quantization of a wave field imparts to it some particle properties; in the case of the electromagnetic field, a theory of light quanta (photons) results. The field quantization technique can also be applied to a field, such as that described by the nonrelativistic Schrodinger equation (6.16) or by one of the relativistic equations (42.4)or (43.3). As we shall see (Sec. 46), it then converts a one-particle theory into a many-particle theory; in the non relativistic case, this is equivalent to the transition from Eq. (6.16) to (16.1) or (32.1). Because of this equivalence, it might seem that the quantization of fields merely provides another formal approach to the many-particle problem. However, the new formalism can also deal as well with processes that involve the creation or destruction of material particles (radioactive beta decay,meson-nucleon interaction). ...
L.I. Schiff, "Quantum Mechanics", McGraw Hill Book Co. New York (1949).

§5.2 Who did the second quantization?

The equivalence of the operator (second-quantized) description and the ordinary Schrodinger theory description of a system of n particles was established by Jordan and Klein [Jordan (1927)] for particles obeying Bose statistics and by Jordan and Wigner [Jordan (1928b)] for particles obeying Fermi statistics. The operator formalism was subsequently re- formulated in " Fock space" by Fock (1932) who gave the generalization of the ordinary Schrodinger wave mechanics to systems for which the number of particles is not a constant of the motion. Fock's work was based on some previous investigations along similar lines by Landau and Peierls [Landau (1930)] who gave a configuration space treatment of the quantized electromagnetic field interacting with matter. Landau and Peierls' treatment was in turn suggested earlier by Oppenheimer, Heisenberg, and Pauli [Heisenberg (1930)] who sketched a configuration space treatment of quantized field 'theories.

Reference: Schweber S. S., *An Introduction to Relativistic Quantum Field Theory*, Row, Peterson And Company (1961).