Lessons in Quantum Field Theory

Quantization of Schrodinger Field

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I Lesson Overview

I.1 Syllabus

Schrodinger quantum mechanics as a classical field theory. Canonical quantization of Schrodinger field.

I.2 Prerequisites

Lagrangian and Hamiltonian Mechanics of systems with finite degrees of freedom; Poisson brackets. Canonical quantization. Schrodinger quantum mechanics;

I.3 Learning Objectives

The objectives of this lesson are

1. To recall the structures of classical mechanics and quantum mechanics.
2. To recall quantization of classical systems.
3. Quantum Mechanics of a point particle
4. To discuss the reinterpretation of one particle quantum mechanics as classical field theory.
5. To learn the classical Lagrangian and Hamiltonian formulation of Schrodinger field.
6. To present canonical quantization of the Schrodinger field.

II Let's Recall and Discuss

II.1 Structure of classical and quantum theories

The essential components of classical and quantum theory of a point particle are compared in the following table.

Form of dynamics	States	Dynamical variables	EOM	Interactions
Lagrangian form	$\mathbf{q}, \dot{\mathbf{q}}$	$F(q_k, \dot{q}_k)$	Euler Lagrange Eqs.	Lagrangian
Hamiltonian Form	$\mathbf{q}, \mathbf{p}$	$F(q_k, p_k)$	Hamilton's Eqs.	Hamiltonian
Poisson bracket form	$\mathbf{q}, \mathbf{p}$	$F(q_k, p_k)$	Poisson bracket Eqs.	Hamiltonian
Quantum Mechanics	$\lvert\psi\rangle$	Hermitian Operators	Schrodinger Eq.	Hamiltonian Op.

II.2 Quantization of Classical Systems

The quantization of a classical system consists in identifying the commutators of $\mathbf{q}\, s$ and $\mathbf{p}\, s$ with $i\hbar\times$ Poisson bracket

$$\big[q_j, p_k\big]_- = \big\{q_j, p_k\big\}_{\text{PB}} = i\hbar\delta_{jk} \tag{1}$$

II.3 Quantum mechanics of a Point Particle

In classical mechanics the states of a point particle are described by generalized coordinates and canonical momenta.

When we come to quantum description of a particle, we no longer describe the states of a point particle by its coordinates and momentum. The dynamical variables position and momentum become operators. The quantum states are described by its wave function $\psi(x,t)$. The wave function obeys the Schrodinger equation

$$i\hbar\frac{\partial\psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi(x,t) + V(x)\psi(x,t). \tag{2}$$

This is the equation of motion of the quantum particle.

III Main Topics

III.1 Reinterpretation of Schrodinger Equation as Field Equation

The state of a particle in quantum mechanics is described by wave function $\psi(\mathbf{x})$. The Schrodinger wave function $\psi(x)$ describes the quantum mechanical states of a point particle. For this system the wave function can be thought of as playing the role of generalized coordinates q_k — the discrete index k getting replaced by a continuous index x. Instead of one generalized coordinate q_k for each k, we now have a generalized coordinate $\psi(x)$ for each x which describe the *quantum states* of the particle.

Thus we have a system with infinite degrees of freedom since x takes infinite number of values.

Therefore, quantum mechanics can be reinterpreted as classical field theory. The Schrodinger equation is taken as a classical field equation.

III.2 Classical Schrodinger Field

The Schrödinger equation can be regarded Euler Lagrange equation for The Lagrangian L given by [1]

$$L = \int dx \mathscr{L}(\psi, \dot{\psi}, \nabla\psi), \tag{3}$$

where $\mathscr{L}$ is the Lagrangian density

$$\mathscr{L} = i\hbar\psi^*(x,t)\frac{\partial\psi(x,t)}{\partial t} - \frac{\hbar^2}{2m}|\nabla\psi|^2 - \psi^*(x,t)V(x)\psi(x,t) \tag{4}$$

The Euler Lagrange equation

$$\frac{\delta L}{\delta\psi(x,t)} = \frac{\partial\mathscr{L}}{\partial\psi} - \frac{d}{dt}\frac{\partial\mathscr{L}}{\partial\dot{\psi}} = 0, \tag{5}$$

for the Lagrangian (3), coincides Verify! with the Schrödinger equation.

[1] We consider Schrödinger equation for a particle in one dimension.

III.3 Canonical Momenta, Hamiltonian, Poisson Brackets

The canonical momentum conjugate to $\psi(x,t)$ is

$$\pi(x,t) = \frac{\delta L}{\delta\dot{\psi}(x,t)} = \frac{\partial \mathscr{L}}{\partial\dot{\psi}(x,t)} = i\hbar\psi^*(x,t). \tag{6}$$

The Hamiltonian of the Schrodinger field can now be computed and is found to be

$$\begin{aligned}
\mathcal{H} &= \int d\mathbf{x}\mathscr{H}(\pi(\mathbf{x}),\psi(\mathbf{x}),\nabla\psi(\mathbf{x})) && (7)\\
\mathscr{H} &= \pi(\mathbf{x})\frac{d\psi(\mathbf{x})}{dt} - \mathscr{L} && (8)\\
&= \frac{\hbar^2}{2m}\nabla\psi^*\nabla\psi + \psi^* V\psi \qquad \text{Verify!} && (9)
\end{aligned}$$

The nonzero Poisson bracket for the Schrodinger field is

$$\{\psi(\mathbf{x}),\pi(\mathbf{y})\}_{\text{PB}} = \delta(\mathbf{x}-\mathbf{y}). \tag{10}$$

All other Poisson brackets are zero.

$$\{\psi(\mathbf{x}),\psi(\mathbf{y})\}_{\text{PB}} = \{\pi(\mathbf{x}),\pi(\mathbf{y})\}_{\text{PB}} = 0 \tag{11}$$

III.4 Canonical Quantization of Schrodinger Field

Canonical Quantization

The Lagrangian density for the Schrodinger field is

$$\mathscr{L} = i\hbar\psi^*(x,t)\frac{\partial\psi(x,t)}{\partial t} - \frac{\hbar^2}{2m}|\nabla\psi|^2 - \psi^*(x,t)V(x)\psi(x,t) \tag{12}$$

The canonical momentum density conjugate to $\psi(x)$ is

$$\pi(x,t) = \frac{\delta L}{\delta\dot{\psi}(x,t)} = \frac{\partial \mathscr{L}}{\partial\dot{\psi}(x,t)} = i\hbar\psi^*(x,t). \tag{13}$$

The act of quantization consist in writing down the equal time commutation relations (ETCR) for the canonically conjugate pair of variables.

$$\begin{aligned}
[\psi(x,t),\pi(y,t)] &= i\hbar\delta(x-y), && (14)\\
[\psi(x,t),\psi(y,t)] = 0, &\qquad [\pi(x,t),\pi(y,t)] = 0. && (15)
\end{aligned}$$

These equation generalize the canonical commutation rules (CCR)

$$[q_j,q_k] = 0, \quad [q_j,_k] = i\hbar\delta_{jk}, \quad [p_j,p_k] = 0, \tag{16}$$

for quantization of systems with finite degrees of freedom, to the case of infinite degrees of freedom. So, with canonical momentum given by (13), the quantization rules for the Schrödinger field take the form

$$\begin{aligned}
[\psi(x,t),\psi^\dagger(y,t)] &= \delta(x-y) && (17)\\
[\psi(x,t),\psi(y,t)] = 0 &\qquad [\psi^\dagger(x,t),\psi^\dagger(y,t)] = 0 && (18)
\end{aligned}$$

Note that complex conjugation $\psi^*(x,t)$does not make sense for operators, one must use adjoint $\psi^\dagger(x,t)$.

Equation of motion

Using the ETCR, it is straight forward to verify that the Heisenberg EOM

$$\frac{d\psi(x,t)}{dt} = \frac{1}{i\hbar}[\psi(x,t), H] \tag{19}$$

$$\frac{d\pi(x,t)}{dt} = \frac{1}{i\hbar}[\pi(x,t), H] \tag{20}$$

imply the Schrodinger equation.

Dynamical Variables

The dynamical variables of the second quantized theory are functionals of the filed operator $\psi(x)$. Hamiltonian is an example of one such variable and is given as integral of Hamiltonian density $\mathscr{H}$

$$\mathcal{H} = \int d\mathbf{x}\, \mathscr{H}(\mathbf{x}) \tag{21}$$

where $\mathscr{H}$ is given by

$$\mathscr{H}(\psi, \psi^\dagger) = \frac{\hbar^2}{2m}(\nabla\psi(\mathbf{x}))^\dagger(\nabla\psi(\mathbf{x})) + \psi^\dagger(\mathbf{x})V(\mathbf{x})\psi(\mathbf{x}) \tag{22}$$

The field momentum $\mathcal{P}$ takes the form

$$\mathcal{P}(\psi, \psi^\dagger) = \int \mathscr{P}\, d\mathbf{x}, \qquad \mathscr{P} = -i\hbar\psi^\dagger(\mathbf{x})\nabla\psi(\mathbf{x}) \qquad \text{How to get this?} \tag{23}$$

The angular momentum of the Schrodinger field is given by

$$\mathscr{M}(\psi, \psi^\dagger) = \int d\mathbf{x}\, \psi^\dagger(\mathbf{x})\mathbf{x} \times \big(-i\hbar\nabla\big)\psi(\mathbf{x}). \qquad \text{How to get this?} \tag{24}$$

The above expressions can be derived using Noether's theorem for time translations, space translations, and space rotations.

Remarks

- ✍ We started with the time dependent Schrodinger equation which is the Schrodinger picture EOM in quantum mechanics. You might have missed a curious fact that we have landed in the Heisenberg picture of the quantized theory.

- ✍ In quantum field theory, you will use both Heisenberg and Dirac pictures only. Schrodinger picture is rarely used in quantum field theory.

- ✍ The cases when the Lagrangian is linear in velocities, the canonical quantization poses a problem. For example, here the canonical momentum conjugate to $\psi^*(x)$ vanishes.

  We follow a standard work around and do not treat field $\psi(x)$ and $\psi^*(x)$ as independent coordinates; $\psi^*(x)$ becomes proportional to the momentum conjugate to $\psi(x)$. This fact should be remembered while using commutation relations.

IV EndNotes

IV.1 Points of Discussion

Why do a second quantization?

Briefly compare the wave mechanics with the second quantized theory paying attention to the following points.

1. Is there a relation between Schrodinger wave mechanics and the second quantized theory?
2. Do the two theories predict the same answers for different processes?
3. Does the second quantized theory offer anything new compared to the wave mechanics?

IV.2 Related Topics

This lesson prepares the ground material for canonical quantization. The act of quantization is complete with writing of canonical commutation rules (CCR) in (14) -(15). The CCR are very powerful and anything that we want to compute, can now be computed.

Before we start computing quantities of physical interest, we need to build the Hilbert space of all states of the system. We also need to get physical interpretation of the mathematical objects and equations we have.

IV.3 Problems for You

1. (*a*) The Lagrangian density for the Schrodinger equation is given to be

$$\mathscr{L} = i\hbar\psi^*(x,t)\frac{\partial\psi(x,t)}{\partial t} - \frac{\hbar^2}{2m}|\nabla\psi|^2 - \psi^*(x,t)V(x)\psi(x,t)$$

   Verify that the Euler Lagrange equations for the Schrodinger field coincide with the Schrodinger equation.

   (*b*) Find the Hamiltonian of the system. Use Poisson brackets to obtain the Hamiltonian equations of motion.

   (*c*) Verify that the Hamilton's equations imply the Euler Lagrange equation of motion.

2. Compute the commutators $[\psi(x), H]; [\pi(x), H]$ and verify that the Heisenberg equations of motion coincide with the Schrodinger equation.

3. Taking the case of free Schrodinger field answer the following questions.

   (*a*) Find Heisenberg equations of motion for the operators $a(k,t)$ and $a^\dagger(k,t)$

   (*b*) Solve the equations of motion and expre

   (*c*) $a(k,t)$ and $a^\dagger(k,t)$ as functions of time. Calculate the unequal time commutators

$$\big[a(k,t), a(k',t')\big], \quad \big[a^\dagger(k,t), a^\dagger(k',t')\big], \quad \big[a(k,t), a^\dagger(k',t')\big].$$

(*d*) Use your answers and work out the unequal time commutator

$$[\psi(x,t), \psi^\dagger(x',t')].$$

(*e*) Use your result for unequal time commutator and express $\psi(x_1,t_1)\psi^\dagger(x_2,t_2)$ in a normal ordered form.

IV.4 Some Quotes

We end this section with a quote from Landau and Lifshitz [**?**]

Classical Quantum Connection

Landau and Lifshitz in their book on Non-relativistic Quantum Mechanics write

> Thus quantum mechanics occupies a very unusual place among physical theories: it contains classical mechanics as a limiting case, yet at the same time it requires this limiting case for its own formulation.

Second Quantization

The reason for introducing the language of second quantization is that it turns out to be extremely convenient in the formulation of a quantum theory for many interacting particles. The starting point of this chapter is the more familiar first quantized N-body Schrodinger equation in the place representation, where the Hamiltonian of interest is motivated from the study of ultracold atomic quantum gases. However, the resulting Hamiltonian is actually seen to be much more general, such that it also applies to a large class of condensed-matter problems.

Reference Henk T.C., StoofKoos B., GubbelsDennis B.M. and Dickerscheid, *Ultracold Quantum Fields*, Springer Netherlands (2009)

What is Second Quantization?

There is a statement by E. Nelson which reads something like this
"Second quantization is a functor, first quantization is a mystery"
– Valter Moretti

Reference: https://physics.stackexchange.com/questions/330428/first-quantization-vs-second- quantization