ME-02 Lectures in Mechanics[*]

Rotation of Coordinate Axes

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Let K be a set of coordinate axes OX_1, OX_2, OX_3. Let K' be another set obtained from K by applying a rotation by an angle θ about an axis passing through the origin. Let the unit vector along the axis of rotation be given by $\hat{n}$. A positive rotation is given by the right hand thumb rule. Hold the axis in right hand, with the thumb pointing along the unit vector $\hat{n}$. The positive direction of rotations is the one in which the fingers curl.

Let the coordinates of a point be (x_1, x_2, x_3) and $(x_1{}', x_2{}', x_3')$ w.r.t. the two sets of axes K and K'. Then the components of the position vector are related by

$$x' = R_{\hat{n}}(\theta)x, \tag{1}$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \text{and} \quad x' = \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} \tag{2}$$

and $R_{\hat{n}}(\theta)$ is a 3×3 matrix and will be called rotation matrix. For rotations about the

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coordinate axes we have the rotation matrices.

$$R_1(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix} \tag{3}$$

$$R_2(\alpha) = \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix} \tag{4}$$

$$R_3(\alpha) = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{5}$$

Remarks

1. We will use a vector notation to collectively denote the components of position vector w.r.t. a chosen set of axis. So for example we will have

$$\vec{x} = (x_1, x_2, x_3), \quad \text{and} \quad \vec{x}\,' = (x_1{}', x_2{}', x_3'). \tag{6}$$

2. For a point on the axis of rotation, the components of the position vector of the point do not change. So

$$x' = R_{\hat{n}}(\theta)x = x, \tag{7}$$

   if x lies on the axis of rotation.

3. A unit vector $\hat{n} = (n_1, n_2, n_3)$ specifies the axis of rotation, θ gives the angle of rotation. Therefore, the number of independent parameters needed to specify a rotation is there because the parameters $n1, n_2, n_3$ satisfy a relation ${n_1}^2 + {n_2}^2 + {n_3}^2 = 1$.

4. Instead of rotating coordinate axes by and angle θ, *one may rotate the vectors by an angle $-\theta$ and keep the axes fixed.* The components of the 'new' vector $\vec{A}\,'$, obtained by rotating a vector $\vec{A}$, will be related by the matrix $R_{\hat{n}}(-\theta)$

$$A' = R_{\hat{n}}(-\theta)A \tag{8}$$

5. A rotation takes right(left) handed system of coordinate axis to another right(left) handed coordinate axes. The converse of this statement is also true. Any two right(left) handed systems of coordinate axes K_1 and K_2 are related by a rotation about some axis.

An explicit form of rotation matrix: Let K and K' be two sets of coordinate axes such that

- l_1, m_1, n_1 are direction cosines of the old X_1 axis w.r.t. the new axes K'.
- l_2, m_2, n_2 are direction cosines of the old X_2 axis w.r.t. the new axes K'.
- l_3, m_3, n_3 are direction cosines of the old X_3 axis w.r.t. the new axes K'

For a point at unit distance from the origin and lying on the old axis $0X_1$, we have

$$x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } x' = \begin{bmatrix} l_1 \\ m_1 \\ n_1 \end{bmatrix} \Longrightarrow \begin{bmatrix} l_1 \\ m_1 \\ n_1 \end{bmatrix} = R \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \tag{9}$$

Similarly, by considering points on $0X_2$ and $0X_3$ we get

$$\begin{bmatrix} l_2 \\ m_2 \\ n_2 \end{bmatrix} = R \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} l_3 \\ m_3 \\ n_3 \end{bmatrix} = R \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{10}$$

Combining (9)-(10) we can write

$$R = \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix}. \tag{11}$$

Properties of rotation matrix

1. The matrix R as obtained above has the following properties which may be verified directly using the properties of direction cosines..The matrix R is an orthogonal matrix
$$R^T R = RR^T = I \Longrightarrow R^{-1} = R^T. \tag{12}$$
Here R^T denotes the transpose of the matrix R. Also $\det R = \vec{e}_1 \cdot (\vec{e}_2 \times \vec{e}_3)$ is the triple product $\vec{e}_1 \cdot (\vec{e}_2 \times \vec{e}_3)$ along the old axes, using their components in K'. Since the axes K and K' are right handed,
$$\det R = \vec{e}_1 \cdot (\vec{e}_2 \times \vec{e}_3) = 1. \tag{13}$$
A matrix O is called an orthogonal matrix if $O^T = O^{-1}$. Thus every rotation may be represented by an orthgonal matrix R with $\det R = +1$. A three by three orthogonal matrix will have determinant ± 1. The rotations correspond to the case $\det R = 1$. The case of $\det R = -1$ corresponds to a rotation followed by an inversion $\vec{x}' = \vec{x}$.

2. The orthogonality property $R^T R = I$, $R \cdot R^T = I$ may also be written as
$$\sum_k R_{ik} R_{jk} = \delta_{ij}, \qquad \sum_k R_{ki} R_{kj} = \delta_{ij}. \tag{14}$$

3. The components of a vector $\vec{A}$ in two different frames are related by

$$\begin{bmatrix} A_1' \\ A_2' \\ A_3' \end{bmatrix} = R \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} \tag{15}$$

which will also be written as $A' = RA$, or,

$$A_l' = \sum R_{lm} A_m \tag{16}$$

4. If we have two vectors $\vec{A}$ and $\vec{B}$, their components will depend on the choice of coordinate axes. But the scalar product $\vec{A}\cdot\vec{B}$ computed from $A_1B_1 + A_2B_2 + A_3B_3$ is the same as computed from $A_1'B_1' + A_2'B_2' + A_3'B_3'$

   To prove the above statement we consider

$$\sum_k A_k' B_k' = \sum_k \sum_i (R_{ki} A_i) \sum_j (R_{kj} B_j) = \sum_{i,j} \sum_k (R_{ki} R_{kj}) A_i B_j \tag{17}$$

$$= \sum_{i,j} \delta_{ij} A_i B_j = \sum_i A_i B_i, \tag{18}$$

$$\therefore \qquad A_1'B_1' + A_2'B_2' + A_3'B_3' = \sum_i A_i B_i = A_1B_1 + A_2B_2 + A_3B_3, \tag{19}$$

   where in the step from Eq.(17) to Eq.(18) the orthogonality property Eq.(14) has been used. The above result can be stated as a property of rotations that under a rotation scalar product of two vectors remains invariant. This implies that the length of a vector $\|A\|(= \sqrt{(A\cdot A)}$, and the angle between two vectors,

$$\cos\theta = \frac{\vec{A}\cdot\vec{B}}{\|\vec{A}\|\,\|\vec{B}\|}, \tag{20}$$

   are also invariant. Similarly, it can be proved that the volume of a parallelopiped with edges, given by $\vec{A}, \vec{B}, \vec{C}$,

$$V = |\vec{A}\cdot\vec{B}\times\vec{C}|$$

   does not change under rotation. These properties make use of the fact that the rotations are represented by orthogonal matrices with determinant $+1$.

5. Conversely, every orthogonal matrix R with $\det R = +1$ represents a rotation. Knowing R and a set of axis K, the new set of axes K' is easily found; the three columns of R give the components of unit vectors along the three new axis.

6. Let S be a 3×3 matrix such that $\sum {A'_k}^2 = \sum {A_k}^2$ holds for every vector $\vec{A}$, where

$$\begin{bmatrix} A'_1 \\ A'_2 \\ A'_3 \end{bmatrix} = S \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix}. \tag{21}$$

   In other words, if the length of a vector $\vec{A}$ as computed from its components (A_1, A_2, A_3) is equal to that computed from the transformed components (A'_1, A'_2, A'_3), then S is an orthogonal matrix:

$$S^T S = I. \tag{22}$$

   The equality $\det S = 1$ does not follow from Eq.(22), it implies only $\det S = \pm 1$.

7. Since a rotation matrix R is 3×3 orthogonal matrix with determinant 1, the eigenvalues of R are given by $1, e^{i\theta}, e^{-i\theta}$. The eigenvector corresponding to 1 will give the axis of rotation.