MM-13 Lessons in Mathematical Physics

The Generating Function and its Applications

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§1.1 Recurrence Relations for Hermite polynomials

The generating function for Hermite polynomials is

$$\exp(2xt - t^2) = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x). \tag{1}$$

We shall derive recurrence relations using the generating function. Differentiating (1) w.r.t.x we get

$$2t \exp(2xt - t^2) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{d}{dx} H_n(x). \tag{2}$$

Substituting (1) in the l.h.s. of (2) we get

$$2\sum_{n=0}^{\infty} \frac{t^{n+1}}{n!} H_n(x) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{d}{dx} H_n(x). \tag{3}$$

Comparing coefficient of t^m, we get

$$2 \times \frac{1}{(m-1)!} H_{m-1}(x) = \frac{1}{m!} \frac{d}{dx} H_m(x). \tag{4}$$

This gives the recurrence relation

$$\boxed{\frac{d}{dx} H_m(x) = 2m H_{m-1}(x).} \tag{5}$$

To derive a recurrence relation of different type, we differentiate (1) w.r.t. t and proceed as above. This gives

$$(2x - 2t) \exp(2xt - t^2) = \sum_{n=0}^{\infty} \frac{nt^{n-1}}{n!} H_n(x) = \sum_{n=1}^{\infty} \frac{t^{n-1}}{(n-1)!} H_n(x). \tag{6}$$

Replacing the exponential in the left hand side by $\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x)$, we get

$$(2x - 2t) \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x) = \sum_{n=1}^{\infty} \frac{t^{n-1}}{(n-1)!} H_n(x) \tag{7}$$

Replacing dummy index $n \to m + 1$, and $m \to n$, we get

$$(2x - 2t) \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x) = \sum_{m=0}^{\infty} \frac{t^n}{n!} H_{n+1}(x). \tag{8}$$

or

$$2x \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x) - 2 \sum_{n=0}^{\infty} \frac{t^{n+1}}{n!} H_n(x) = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_{n+1}(x). \tag{9}$$

Comparing coefficients of t^m on both sides, we get

$$\frac{2x}{m!} H_m(x) - \frac{2}{(m-1)!} H_{m-1}(x) = \frac{1}{m!} H_{m+1}(x). \tag{10}$$

This gives us the following recurrence relation

$$\boxed{H_{m+1}(x) = 2x H_m(x) - 2m H_{m-1}(x).} \tag{11}$$

§2 Othogonality and Normalization of Hermite Polynomials

§2.1 Orthogonality and Normalization

We shall use the generating function of Hermite polynomials to prove the orthogonality and to compute the normalization integral of Hermite polynomials

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x) dx = \sqrt{\pi} 2^n n! \delta_{mn}. \tag{12}$$

We take the product of two generating functions

$$\exp(2xt - t^2) = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x). \tag{13}$$

and

$$\exp(2xs - s^2) = \sum_{n=0}^{\infty} \frac{s^n}{m!} H_m(x). \tag{14}$$

and write

$$\exp(2xt - t^2)\exp(2xs - s^2) = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x) \times \sum_{n=0}^{\infty} \frac{s^m}{m!} H_m(x). \tag{15}$$

or

$$\sum_n \sum_m \frac{t^n}{n!}\frac{s^m}{m!} H_n(x) H_m(x) = \exp(2tx + 2sx - s^2 - t^2). \tag{16}$$

Next multiply (17) by e^{-x^2} and integrate over x from $-\infty$ to ∞.

$$\sum_n \sum_m \frac{t^n}{n!}\frac{s^m}{m!} \int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x)\, dx = \int_{-\infty}^{\infty} e^{-x^2} \exp(2tx + 2sx - s^2 - t^2)\, dx. \tag{17}$$

The right hand side then becomes

$$\begin{aligned}
&\int_{-\infty}^{\infty} \exp(2tx + 2sx - t^2 - s^2)\exp(-x^2) dx \\
&= \exp(-s^2 - t^2) \int_{-\infty}^{\infty} \exp\ [2x(s+t) - x^2]\ dx \\
&\qquad \text{Completed the square in the exponential} \\
&= \exp(-s^2 - t^2)\exp[(s+t)^2)] \int_{-\infty}^{\infty} \exp[-(x - s - t)^2] \\
&= \exp(2st) \int_{-\infty}^{\infty} e^{-x^2}\, dx \\
&= \sqrt{\pi}\, e^{2st}
\end{aligned} \tag{18}$$

Using (17) and (18), we get

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{t^n}{n!}\frac{s^m}{m!} \int_{-\infty}^{\infty} H_n(x) H_m(x) e^{-x^2}\, dx = \sqrt{\pi} e^{2st}. \tag{19}$$

The right hand side is now expanded in powers of s and t and we compare the coefficients of $t^n s^m$ on both sides. Notice that the right hand side will not have any term $t^n s^m$ in which the powers n and m are different $m \neq n$. This gives

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x) dx = 0, \qquad \text{for } m \neq n. \tag{20}$$

Expand the exponential in the right hand side of (19) and write it as

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{t^n}{n!}\frac{s^m}{m!}\int_{-\infty}^{\infty} H_n(x)H_m(x)e^{-x^2}\,dx = \sqrt{\pi}\sum_{n=0}^{\infty}\frac{(2st)^n}{n!}. \tag{21}$$

Comparing the coefficients of $(st)^n$ on both sides of this equation we get

$$\frac{1}{n!}\frac{1}{n!}\int_{-\infty}^{\infty} H_n(x)H_n(x)e^{-x^2}\,dx = \frac{\sqrt{\pi}2^n}{n!}. \tag{22}$$

Thus we get the normalization integral to be

$$\int_{-\infty}^{\infty} H_n(x)H_n(x)e^{-x^2}\,dx = \sqrt{\pi}2^n n!. \tag{23}$$

The equations (20) and (23) can be combined into a single equation as

$$\boxed{\int_{-\infty}^{\infty} H_n(x)H_m(x)e^{-x^2}\,dx = \sqrt{\pi}\,2^n\,n!\,\delta_{mn}.} \tag{24}$$

§3 Normalization for Legendre Polynomials

Working with example of Legendre polynomials, and assuming the orthogonality property, we shall obtain the normalization for the Legendre polynomials

$$\int_{-1}^{1} dx\,P_\ell(x)P_\ell(x) = \frac{2}{2\ell+1}. \tag{25}$$

The generating function for Legendre polynomials is

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} t^n P_n(x). \tag{26}$$

We take one more generating function of the Legendre polyonomials

$$\frac{1}{\sqrt{1-2xs+s^2}} = \sum_{n=0}^{\infty} s^m P_m(x). \tag{27}$$

and multiply the two expressions and integrate over x from -1 to 1. This gives

$$\int_{-1}^{1} dx\,\frac{1}{\sqrt{1-2xt+t^2}}\frac{1}{\sqrt{1-2xs+s^2}} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} s^n t^m \int_{-1}^{1} P_n(x)P_m(x)\,dx. \tag{28}$$

If we are required to prove orthogonality, we must proceed to integrate the l.h.s. which will turn out to be a function of the product (st). Hence the terms with different powers of $t^n s^m, m \neq n$ will be absent, and this will imply orthogonality of Legendre polynomials. However we will skip details of this.

To obtain the normalization integral (25), we set $s = t$ in (28), to get

$$\int_{-1}^{1} \frac{1}{1-2xt+t^2} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} t^{m+n} \int_{-1}^{1} P_n(x)P_m(x)\,dx \tag{29}$$

In the right hand side the terms with $m \neq n$ are zero because of orthogonality property, hence we get

$$\sum_{n=0}^{\infty} t^{2n} \int_{-1}^{1} P_n(x)P_n(x)\,dx = \int_{-1}^{1} \frac{dx}{1-2tx+t^2}. \tag{30}$$

Integrating over x gives

$$\begin{aligned}
\sum_{n=0}^{\infty} t^{2n} \int_{-1}^{1} P_n(x)P_n(x)\,dx &= -\frac{1}{2t}\log(1-2tx+t^2)\Big|_{-1}^{1} \\
&= -\frac{1}{2t}\{\log(1-t)^2 - \log(1+t)^2\} \\
&= \frac{1}{t}\{\log(1+t) - \log(1-t)\} \\
&= 2\Big[1 + \frac{t^2}{3} + \frac{t^4}{5} + \ldots \frac{t^{2n}}{(2n+1)} + \ldots\Big].
\end{aligned} \tag{31}$$

Comparing coefficients of t^{2n} on both sides we get the desired result

$$\boxed{\int_{-1}^{1} P_n(x)P_n(x)\,dx = \frac{2}{2n+1}}. \tag{32}$$

§4 Miscellaneous Applications of Generating Function

We will show that the generating function can be used to get the values of the orthogonal polynomials at special points.

$P_n(1) = 1$ **and** $P_n(-1) = (-1)^n$**:** The generating function for the Legendre polynomials is

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} t^n P_n(x). \tag{33}$$

Setting $x = 1$ gives

$$\sum_{n=0}^{\infty} t^n P_n(1) = \frac{1}{1-t} = 1 + t + t^2 + t^n + . \tag{34}$$

Comparing the coefficient of t^n, we get $P_n(1) = 1$. If We set $x = -1$ in (33), we would get

$$\sum_{n=0}^{\infty} t^n P_n(-1) = \frac{1}{1+t} = 1 - t + t^2 + (-1)^n t^n + \ldots \tag{35}$$

giving $P_n(-1) = (-1)^n$.

$L_n(0) = 1$**:** The generating function for Laguerre polynomials is

$$\sum_{n=0}^{\infty} t^n L_n(x) = \frac{e^{xt/(1-t)}}{(1-t)}. \tag{36}$$

Setting $x = 0$ gives

$$\sum_{n=0}^{\infty} t^n L_n(0) = \frac{1}{(1-t)} = \sum_{n=0}^{\infty} t^n. \tag{37}$$

Therefore, comparing the coefficients of t^n we get the result $L_n(0) = 1$.

References Dennery P. and Kryzywicki, *Mathematics for Physicists*,